Math 3130 - Assignment 1

Due January 22, 2016
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Solve all systems of linear equations by row reduction (Gaussian elimination).

(1) Do the following 4 planes intersect in a point? Which?

$$\begin{aligned} x + 5y + 3z &= 16 \\ 2x + 10y + 8z &= 34 \\ 4x + 20y + 15z &= 67 \\ x + 6y + 5z &= 21 \end{aligned}$$

Solution:

$$\sim \begin{bmatrix} 1 & 5 & 3 & 16 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 & 16 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 & 13 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The planes intersect in the point $(x_1, x_2, x_3) = (-2, 3, 1)$. □

(2) Add an equation of a line to the equation

$$2x + 3y = 4$$

such that the resulting system has (a) no solution, (b) exactly one solution, (c) infinitely many solutions.
Solution:
(a) $0 = 1$, (b) $y = 1$, (c) $2x + 3y = 4$. □

(3) For which values $a \in \mathbb{R}$ does the following system of linear equations have more than one solution?

$$\begin{aligned} x + 2y &= 0 \\ 2x + ay &= 0 \end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & a & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & a-4 & 0 \end{bmatrix}$$

If $a = 4$, then x_2 is a free variable and we have infinitely many solutions. If $a \neq 4$, then there is no free variable and we have one solution. □

(4) First transform to echelon form, then to reduced echelon form. Solve the corresponding system of linear equations.

$$\begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix}$$

Solution:

$$\sim \begin{bmatrix} 2 & -4 & 1 & 0 \\ 0 & 0 & 2 & 4 \\ -3 & 6 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 7 & 14 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution in parametric form: $(x_1, x_2, x_3) = (-1 + 2r, r, 2)$. □

(5) First transform to echelon form, then to reduced echelon form. Solve the corresponding system of linear equations.

$$\begin{bmatrix} 2 & 2 & 1 & 8 & 2 \\ 2 & 0 & 0 & 8 & 2 \\ 2 & 6 & 3 & 8 & 1 \end{bmatrix}$$

Solution:

$$\sim \begin{bmatrix} 2 & 2 & 1 & 8 & 2 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 & 8 & 2 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

From the echelon matrix we see that there is no solution. We convert into reduced echelon form:

$$\sim \begin{bmatrix} 2 & 0 & 0 & 8 & 2 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

□

(6) First transform to echelon form, then to reduced echelon form. Solve the corresponding system of linear equations.

$$\begin{bmatrix} 0 & 0 & 3 & 1 & 0 \\ 1 & 2 & 2 & -1 & 1 \\ 0 & 4 & 5 & -2 & 0 \\ 2 & 0 & 2 & 1 & 2 \\ -1 & 2 & 6 & 0 & -1 \end{bmatrix}$$

Solution:

$$\sim \begin{bmatrix} 1 & 2 & 2 & -1 & 1 \\ 2 & 0 & 2 & 1 & 2 \\ -1 & 2 & 6 & 0 & -1 \\ 0 & 4 & 5 & -2 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & -1 & 1 \\ 0 & 4 & 2 & -3 & 0 \\ 0 & 4 & 8 & -1 & 0 \\ 0 & 4 & 5 & -2 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & -1 & 1 \\ 0 & 4 & 2 & -3 & 0 \\ 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -1 & 1 \\ 0 & 4 & 2 & -3 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -\frac{5}{3} & 1 \\ 0 & 4 & 0 & -\frac{11}{3} & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{6} & 1 \\ 0 & 1 & 0 & -\frac{11}{12} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution in parametric form: $(x_1, x_2, x_3, x_4) = (1 - \frac{1}{6}r, \frac{11}{12}r, -\frac{1}{3}r, r)$. □

(7) [1, Section 1.3, Ex 12] Is $\mathbf{b}$ a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$?

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & -11 & 2 \end{bmatrix}$$

We see that the linear system in echelon form has no row of the form $0 = a$ with $a \neq 0$. Thus the system has a solution, and hence $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. □

(8) [1, Section 1.3, Ex 16] For which values of h is $\mathbf{y}$ in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 2h-5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h+4 \end{bmatrix}$$

For $h = -2$ we obtain one solution, and for $h \neq -2$ no solution. Thus $\mathbf{y}$ is in the plane if and only if $h = -2$. □

(9) Are the following true or false? Explain your answers.
  (a) Any system of linear equations with strictly less equations than variables has infinitely many solutions.
  (b) Different sequences of elementary row reductions can transform one matrix to distinct matrices in row echelon form.
  (c) The vector $3\mathbf{v_1}$ is a linear combination of the vectors $\mathbf{v_1}, \mathbf{v_2}$.
  (d) For $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^3$, Span$\{\mathbf{v_1}, \mathbf{v_2}\}$ is always a plane through the origin.

Solution:

  (a) **False:** There might be no solution at all, like for $x + y + z = 0$ and $0 = 1$.
  (b) **True:** E.g., the matrix $A = [2\ 4]$ can be reduced to echelon form $[2\ 4]$ or $[1\ 2]$.
  (c) **True**

(d) **False:** It might be a plane or a line or just the origin (if $\mathbf{v_1} = \mathbf{v_2} = 0$). E.g., $\text{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\}$ is just the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ because the second vector is a multiple of the first.

$\square$

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.