Non-Parametric Integrals:

- These are standard double and triple integrals where no parametrization is needed.

General Facts:

- Compute as an iterated integral
- For nice f, we can change the order of integration.
- The final output of the integral should be a number (NO VARIABLES!)
- Jacobians show up when we change coordinates ($r\,dr\,d\theta$, $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, etc.)

Double Integrals:

- Usually use cartesian (x,y) or polar (r,θ) coordinates.
- Common applications:

  Let R be a region in the xy-plane.

  - $\iint_R 1\, dA$ gives the area of R.
  - $\iint_R f(x,y)\, dA$ gives the volume of the 3D region above R below $f(x,y)$.

    If $f(x,y) < 0$, this counts as negative volume.
  - If $\rho(x,y)$ gives the density of a lamina at (x,y),

    $M = \iint_R \rho(x,y)\, dA$ is the mass of the lamina,

    $M_x = \iint_R y\,\rho(x,y)\, dA$ is the moment of mass about the x-axis,

    $M_y = \iint_R x\,\rho(x,y)\, dA$ is the moment of mass about the y-axis,

    and $\left(\frac{M_y}{M}, \frac{M_x}{M}\right) = (\bar{x}, \bar{y})$ is the centroid of the lamina.
- Note: $\iint_R 1\, dA$ gives an area, but $\iint_R f(x,y)\, dA$ gives a volume, which makes sense (ignoring units) because the area of R and the volume of the solid above R with height 1 are the same.

Triple Integrals:

- Usually cartesian (x,y,z), spherical (ρ,ϕ,θ), or cylindrical (r,θ,z) coordinates.
- Common applications:
  - $\iiint_E 1\, dV$ gives the volume of E.
  - For a density $\rho(x,y,z)$, $\iiint_E \rho(x,y,z)\, dV$ gives the mass of the solid E.
    - Centroids can also be found like in 2-dimensions.

Tips:

- If possible, draw your region to help change the order of integration.
- Memorize the common Jacobians to save time.

Parametric Integrals:

- Includes line and surface integrals
- Both require a choice of parametrization.

Line Integrals:

- We want to integrate a function or vector field along a curve C.

Scalar Line Integrals:

- An example to keep in mind is $\int_C \rho(x,y,z)\,ds$, where C is the path traced out by a wire and $\rho(x,y,z)$ is the density of the wire at (x,y,z). Then $\int_C \rho(x,y,z)\,ds$ gives the mass of the wire. Like in Calc I, we can think of scalar line integrals as "adding up" a function along a curve.
- Computing $\int_C f\,ds$:
  1. Parametrize C as $\vec{r}(t) = \langle x(t), y(t)\rangle$ or $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ with $a \le t \le b$ (Bounds are required!). There are usually several reasonable choices for $\vec{r}(t)$.
  2. Find $|\vec{r}\,'(t)|$ and $f(\vec{r}(t)) = f(x(t), y(t), z(t))$.
  3. Evaluate $\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$.

Vector Line Integrals:

- An example to keep in mind is $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y,z)$ is the force at (x,y,z) (force has a magnitude and a direction, so it is a vector) and C is a path. Then $\int_C \vec{F}\cdot d\vec{r}$ is the work done by travelling along C. We can think of $\int_C \vec{F}\cdot d\vec{r}$ as adding up the amount of $\vec{F}$ pointing in the same direction as C along C.
- Computing $\int_C \vec{F}\cdot d\vec{r}$:
  1. Parametrize C as $\vec{r}(t) = \langle x(t), y(t)\rangle$ or $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ with $a \le t \le b$ (Bounds are required!). There are usually several reasonable choices for $\vec{r}(t)$.
  2. Find $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t), z(t))$ and $\vec{r}\,'(t)$. Evaluate
     $$\underbrace{\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\text{Scalar function of } t.}$$
  3. Evaluate $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$.

Surface Integrals:

- We want to integrate a scalar function or vector field over a surface S in 3-dimensional space.

Scalar Surface Integrals:

- $\iint_S f(x,y,z)\, dS$ "adds up" f over the surface S.
- Computing $\iint_S f(x,y,z)\, dS$:
  ① Parametrize S as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ with u, v in a bounded region D (You must specify D!)
  ② Find $f(\vec{r}(u,v)) = f(x(u,v), y(u,v), z(u,v))$, $\vec{r}_u(u,v)$, $\vec{r}_v(u,v)$, and $|\vec{r}_u \times \vec{r}_v|$.
  ③ Evaluate $\iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$, which is a regular double integral.

Vector Surface Integrals:

- An example to keep in mind is $\iint_S \rho(x,y,z)\, \vec{V}(x,y,z) \cdot d\vec{S}$, where $\rho(x,y,z)$ is the density of a fluid at (x,y,z) and $\vec{V}(x,y,z)$ is the velocity of the fluid at (x,y,z). Then $\iint_S \rho\vec{V} \cdot d\vec{S}$ is the rate of flow of liquid through the surface S. We can think of $\iint_S \vec{F} \cdot d\vec{S}$ as adding up the amount of $\vec{F}$ perpendicular to S.
- In order to compute $\iint_S \vec{F} \cdot d\vec{S}$, we need a way to decide what is positive flow and what is negative flow. This is called an orientation. This is a choice of a unit normal vector at every point of S. Unless otherwise stated, assume outward is positive for closed surfaces and up is positive for not closed surfaces.
- Computing $\iint_S \vec{F} \cdot d\vec{S}$:
  ① Parametrize S as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for (u,v) in a domain D (You must state what D is!)
  ② Compute $\vec{V} = \vec{r}_u \times \vec{r}_v$. If $\vec{V}$ points in the same direction as your orientation, good. If not, use $\vec{V} = -(\vec{r}_u \times \vec{r}_v)$. (Always check the orientation!)
  ③ Compute $\vec{F}(\vec{r}(u,v)) = \vec{F}(x(u,v), y(u,v), z(u,v))$ and $\vec{F}(\vec{r}(u,v)) \cdot \vec{V}$.
  ④ Evaluate $\iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{V}\, dA = \pm \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$.

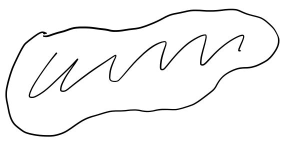
Integral Theorems:

- All of the integral theorems will have roughly the following form:

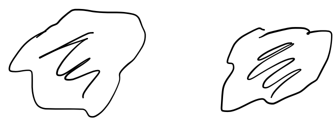
Integral over a curve or a surface or a volume $\xrightarrow{\text{"undo a derivative"}}$ Integral over the boundary of a curve (two points) or a surface (a curve) or a volume (a surface)

Integral over the boundary of a curve (two points) or a surface (a curve) or a volume (a surface) $\xrightarrow{\text{"Take a derivative"}}$ Integral over a curve or a surface or a volume

- A region is called bounded if it can be put into a big enough box.
- A region is connected if it only has one piece.
- A region is simply connected if it only has one piece and has no "holes."



Simply Connected



Not connected

Connected, but not simply connected

Theorems for Line Integrals:

- A vector field $\vec{F}$ is called conservative if $\vec{F} = \nabla f$ for some f. We call f a potential function for $\vec{F}$.

$$\vec{F} \text{ conservative} \Rightarrow \begin{cases} \operatorname{curl} \vec{F} = \vec{0}, \quad \oint_C \vec{F}\cdot d\vec{r} = 0 \text{ for all } C, \\ \vec{F} \text{ is path independent, } \vec{F} \text{ has a potential function.} \end{cases}$$

The other direction is not always true!

If the domain of $\vec{F}$ is open and simply connected, then

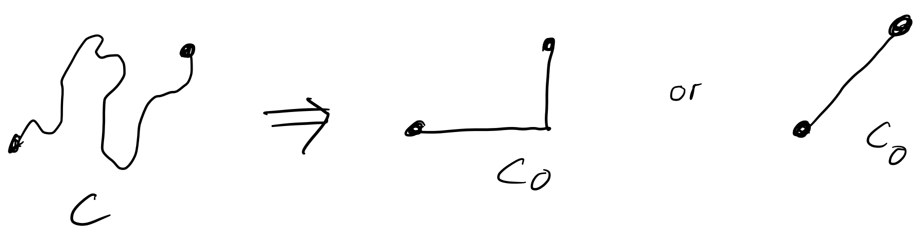
$$\left\{ \begin{array}{l} \vec{F} \text{ conservative} \\ \operatorname{curl} \vec{F} = \vec{0} \\ \oint_C \vec{F}\cdot d\vec{r} = 0 \text{ for all closed } C \\ \quad\text{and} \\ \vec{F} \text{ path independent} \end{array} \right\}$$

are equivalent.

- Uses: • If curl $\vec{F} \neq \vec{0}$, then $\vec{F}$ is <u>not</u> conservative.
  • If the domain of $\vec{F}$ is open and simply connected <u>and</u> curl $\vec{F} = \vec{0}$, then $\vec{F}$ <u>is</u> conservative.

- Why do we care?

- If $\vec{F}$ is conservative and C is a closed curve, then $\oint_C \vec{F} \cdot d\vec{r} = 0$ and we don't have to integrate.
- If $\vec{F}$ is conservative and C is a complicated curve, we can choose a new curve C_0 with the same endpoints and $\int_C \vec{F} \cdot d\vec{r} = \int_{C_0} \vec{F} \cdot d\vec{r}$. Hopefully the new integral will be easier.



The Fundamental Theorem of Line Integrals: (2D and 3D)

- If $\vec{F}$ is conservative and we can find a potential function f, that is, $\vec{F} = \nabla f$, then for a curve C with starting point a and ending point b,

$$\int_C \vec{F} \cdot d\vec{r} = f(b) - f(a)$$

Curve: C $\xrightarrow{\text{undo a gradient}}$ Boundary of C: a and b

Take a gradient and fill in C.

Integral Theorems Involving Surfaces and Volumes:

Green's Theorem: (2D)

If C is a simple closed curve in the plane and D is the region surrounded by C, then for functions $P(x,y)$ and $Q(x,y)$,

$$\iint_D Q_x - P_y \, dA = \oint_C P\,dx + Q\,dy$$

undo partials

Region in the plane: D

The boundary of D: C

Fill in D and take partials.

Note: Both theorems have extra conditions. Read the book for exact statements.

Stokes Theorem:

- If S is an oriented surface with simple, closed, positively oriented boundary curve C, and $\vec{F}$ is a nice vector field, then

$$\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

$$\begin{tikzcd}
\text{Surface: } S \arrow[r, bend left, "\text{``undo a curl''}"] & \text{Boundary of } S\text{: } C \arrow[l, bend left, "\text{Fill in a surface and take a curl.}"]
\end{tikzcd}$$

- A tricky use:

  If S_1 is a surface with boundary curve C and S_2 is another surface with the same boundary, then

$$\iint_{S_1} \text{curl}\, \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \text{curl}\, \vec{F} \cdot d\vec{S},$$

  so, if we know our vector field is the curl of something, we can change surfaces as long as the boundary is fixed.

Divergence Theorem:

- If E is a simple solid region with boundary surface S, which we orient outward, and if $\vec{F}$ is a "nice" vector field, then

$$\iiint_E \text{div}\, \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

$$\begin{tikzcd}
\text{Region: } E \arrow[r, bend left, "\text{``undo a divergence''}"] & \text{Boundary of } E\text{: } S \arrow[l, bend left, "\text{Fill in } E \text{ and take divergence.}"]
\end{tikzcd}$$