University of Colorado Boulder

Math 2400, Midterm 3

Spring 2017

PRINT YOUR NAME: Solutions

PRINT INSTRUCTOR'S NAME: ______________________

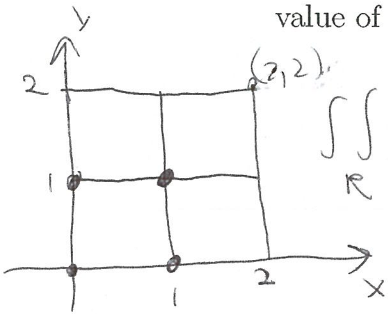
SECTION #: ______________________

Question	Points	Score
1	13	
2	16	
3	8	
4	8	
5	15	
6	8	
7	16	
8	16	
Total:	100	

- No calculators, cell phones, or other electronic devices may be used at any time during the exam.
- Show all of your reasoning and work for full credit, unless indicated otherwise. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like $\ln(3)/2$ as is.
- For multiple choice questions, circle the correct answer.
- Notation: following the book, we use boldface to denote vectors, e.g., $\mathbf{a}, \mathbf{b}$ are vectors.

1. (13 points) Short answer questions.

(a) (5 pts.) Let $R = [0,2] \times [0,2]$. Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R x + y \, dA$. Take the sample points to be the lower left corners.



$$\iint_R (x+y)\,dA \approx (0+0)\cdot 1 + (1+0)\cdot 1 + (0+1)\,1 + (1+1)\cdot 1$$
$$= 0 + 1 + 1 + 2 = 4$$

(b) (3 pts.) Let the density function of a lamina D be $\rho(x, y) = x^2 y$. Use a double integral to write a formula for the mass of the lamina D.

$$m = \iint_D x^2 y \, dA$$

(c) (5 pts.) Let $\mathbf{F} = \langle x, y, z \rangle$, and C be a curve parameterized by $\mathbf{r}(t) = \langle t, e^t, \sin t \rangle$, $0 \le t \le 1$. Set up, but DO NOT evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$r'(t) = \langle 1, e^t, \cos t \rangle$

$$\int_{\boxed{0}}^{\boxed{1}} \boxed{\langle t, e^t, \sin t \rangle} \cdot \boxed{\langle 1, e^t, \cos t \rangle} \, d\boxed{t}$$

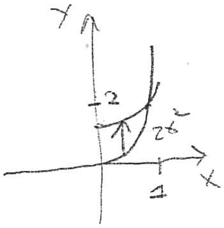
$$= \int_{\boxed{0}}^{\boxed{1}} \boxed{t} + \boxed{e^{2t}} + \boxed{\sin t \cos t} \, d\boxed{t}$$

2. (16 points) (a) (8 pts.) Evaluate the double integral,

$$\iint_D x \, dA,$$

where,

$$D = \{(x, y) : 0 \leq x \leq 1,\ 2x^2 \leq y \leq 1 + x^2\}.$$



$2x^2 = 1 + x^2$

$x^2 = 1$

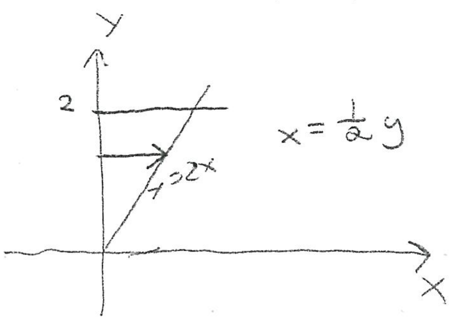
$$\int_0^1 \int_{2x^2}^{1+x^2} x \, dy \, dx$$

$$= \int_0^1 x(1 + x^2 - 2x^2) \, d\cancel{y}x = \int_0^1 x(1 - x^2) \, dx$$

$$= \int_0^1 x - x^3 \, dx = \frac{x^2}{2} - \frac{x^4}{4} \Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{4}}$$

(b) (8 pts.) Evaluate the following double integral by changing the order of integration.

$$\int_0^1 \int_{2x}^2 e^{y^2} \, dy dx =$$



$x = \frac{1}{2} y$

$$= \int_0^2 \int_0^{\frac{1}{2}y} e^{y^2} \, dx \, dy$$

$$= \int_0^2 \frac{1}{2} e^{y^2} y \, dy$$

$u = y^2$

$du = 2y \, dy$

$\frac{1}{4} du = \frac{1}{2} y \, dy$

$$= \frac{1}{4} \int_0^4 e^u \, du$$

$$= \frac{1}{4}(e^4 - 1)$$

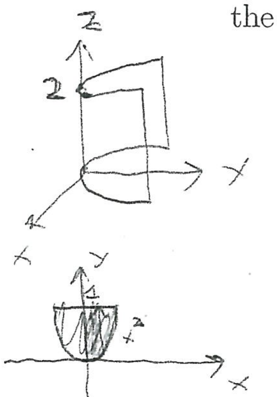
$y = 0 \Rightarrow u = 0$

$y = 2 \Rightarrow u = 4$

3. (8 points) Evaluate the triple intergral

$$\iiint\limits_E y\,dV,$$

where E is the solid in the first octant bounded by the parabolic cylinder $y = x^2$ and the planes $z = 2$, and $y = 1$.



1st octant

$$\int_0^1\int_{x^2}^1\int_0^2 y\,dz\,dy\,dx$$

$$=\int_0^1\int_{x^2}^1 2y\,dy\,dx$$

$$=\int_0^1 y^2\Big|_{x^2}^1\,dx$$

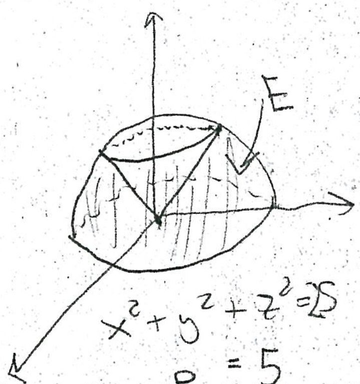
$$=\int_0^1 1-x^4\,dx$$

$$=1-\left(\frac{x^5}{5}\Big|_0^1\right)$$

$$=1-\frac{1}{5}=\boxed{\frac{4}{5}}$$

4. (8 points) Using spherical coordinates, set up an integral which finds the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 25$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$. DO NOT evaluate.

$$\int_{\boxed{\frac{\pi}{4}}}^{\text{pi/2}} \int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{0}}^{\boxed{5}} \boxed{1 \cdot \rho^2 \sin\phi}\, d\boxed{\rho}\, d\boxed{\theta}\, d\boxed{\phi}$$



$x^2+y^2+z^2=25$

$\rho = 5$

cone : $\phi = \frac{\pi}{4}$

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

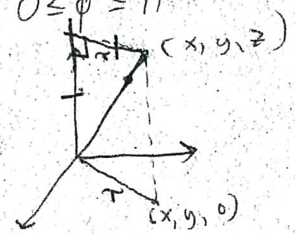
$= \sqrt{\rho^2\sin^2\phi}$

$= \rho|\sin\phi|$

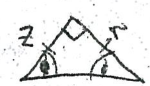
$= \rho\sin\phi \qquad 0 \le \phi \le \pi$

$\Rightarrow \cos\phi = \sin\phi$

$\phi = \frac{\pi}{4}$ or



$z = \sqrt{x^2+y^2} = r$



$\phi = \frac{\pi}{4}$

5. (15 points) Let $a, b > 0$. In the parts that follow, you will show that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ has area πab by making the following change of variables

$$x = ar\cos\theta, \quad y = br\sin\theta.$$

(a) (5 pts.) Compute the Jacobian of the transformation.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix}$$

$$= abr\cos^2\theta + abr\sin^2\theta$$
$$= abr$$

(b) (5 pts.) Under this change of variables, the ellipse is the image of a disk. Find the radius of that disk. (Hint: plug in the change of variables to the equation for the ellipse.)

$$\frac{\cancel{a^2} r^2 \cos^2\theta}{\cancel{a^2}} + \frac{\cancel{b^2} r^2 \sin^2\theta}{\cancel{b^2}} \leq 1$$

$$r^2\cos^2\theta + r^2\sin^2\theta \leq 1$$

$$r^2 \leq 1 \quad \text{so} \quad \boxed{r = 1}$$

(c) (5 pts.) Using parts (a) and (b), apply the change of variables formula for a double integral to compute the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1.$$

$$A(\bigcirc) = \iint_{\bigcirc} 1\, dA = \iint_{r \leq 1} 1 \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\, d\theta$$

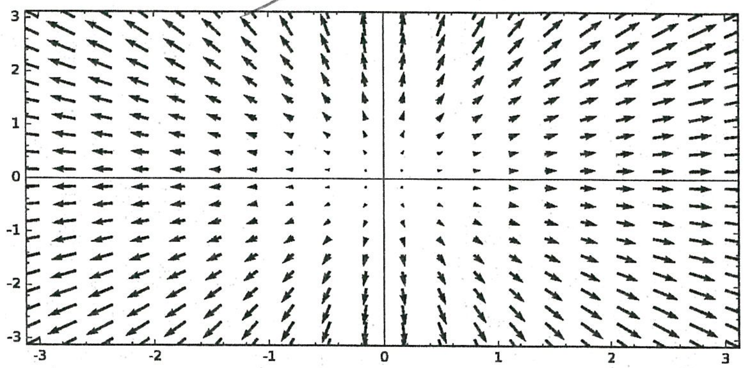
$$= \int_0^{2\pi} \int_0^1 1\, abr\, dr\, d\theta$$

$$= (2\pi ab) \int_0^1 r\, dr = 2\pi ab \left. \frac{r^2}{2} \right|_0^1$$

$$= \boxed{\pi ab}$$

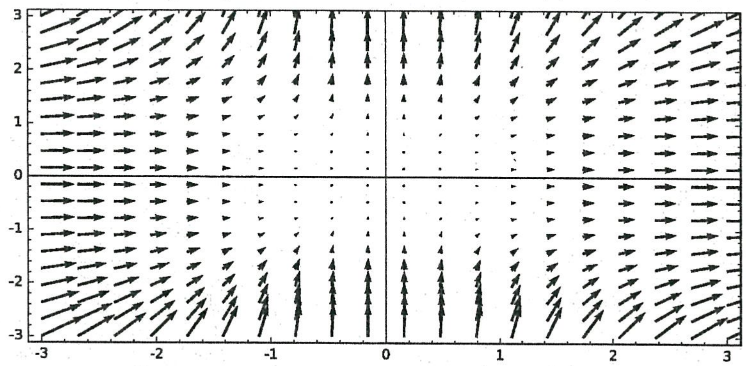
6. (8 points) Match the vector field with the equation describing the vector field. No justification is needed.

1. Vector Field 1: IV



I.) $\mathbf{F}(x,y) = (2-x)\mathbf{i}+(3-y)\mathbf{j}$

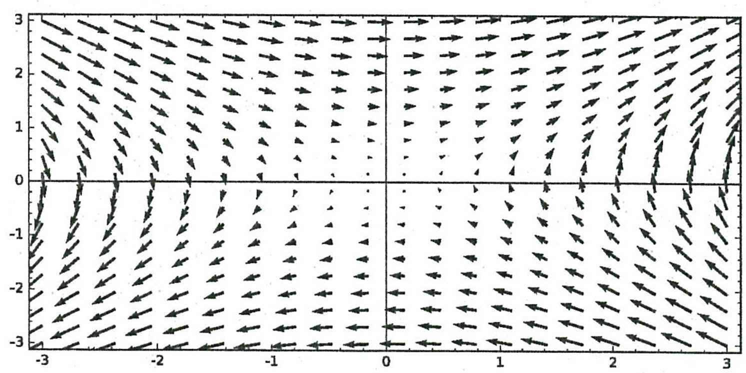
2. Vector Field 2: III



II.) $\mathbf{F}(x,y) = y\mathbf{i} + x\mathbf{j}$

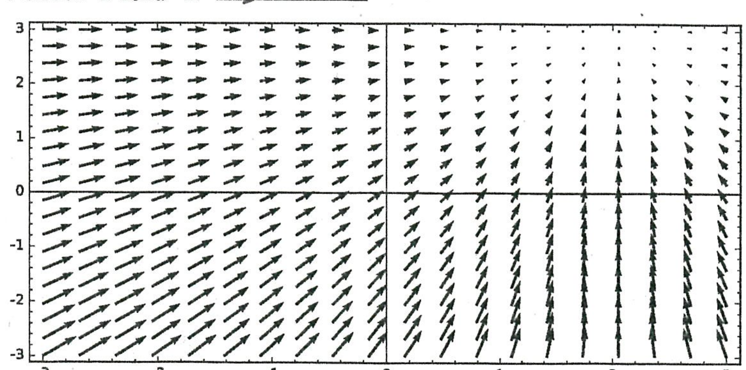
3. Vector Field 3: II

III.) $\mathbf{F}(x,y) = x^2\mathbf{i} + y^2\mathbf{j}$



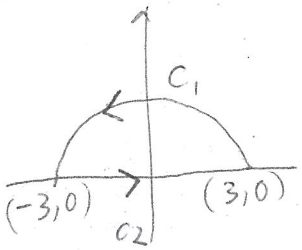
IV.) $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$

4. Vector Field 4: I



7. (16 points) Let C be the curve traced out by traversing the upper half of a circle of radius 3 counterclockwise from $(3,0)$ and then traveling along the line segment from the point $(-3,0)$ to the point $(3,0)$.

(a) (8 pts.) Find a parametrization for C. You may divide C into multiple segments.



$C_1: \langle 3\cos t, 3\sin t\rangle \quad 0 \le t \le \pi$

$C_2: (1-t)\langle -3,0\rangle + t\langle 3,0\rangle = \langle 3t-3, 0\rangle + \langle 3t, 0\rangle$

$= \langle 6t-3, 0\rangle \quad 0 \le t \le 1$

(b) (8 pts.) Calculate $\int_C y\, ds$.

$\int_C y\,ds = \int_{C_1} y\,ds + \int_{C_2} y\,ds$

$= \int_0^{\pi} 3\sin t\, |r'(t)|\, dt + \int_0^1 0\, dt$

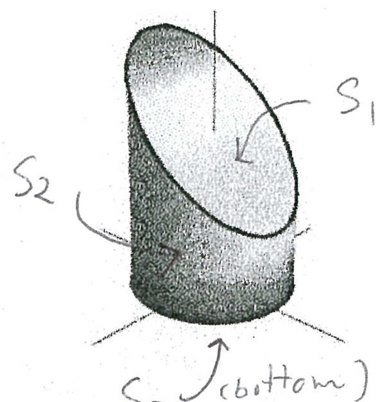
$= \int_0^{\pi} 3\sin t \sqrt{9\cos^2 t + 9\sin^2 t}\, dt$

$= \int_0^{\pi} 9\sin t\, dt$

$= -9\cos t \Big|_0^{\pi}$

$= -9(-1-1) = 18$

8. (16 points) Let S be the surface of the solid obtained by taking a section of the cylinder $x^2 + y^2 = 1$ between the planes $z = 2 - y$ and $z = 0$ as shown in the figure below.



(a) (6 pts.) The upper face of S, the part lying in the plane $z = 2 - y$, may be parametrized by $\mathbf{r}(x, y) = \langle x, y, 2 - y \rangle$, where $(x, y) \in \{(x, y) : x^2 + y^2 \leq 1\}$. Compute the surface area of that portion of the surface S.

$r_x = \langle 1, 0, 0 \rangle \quad r_y = \langle 0, 1, -1 \rangle \qquad r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = i\,0 - j(-1) + k = \langle 0, 1, 1 \rangle$

so $|r_x \times r_y| = \sqrt{2}$

$S_1 = \iint_{O} \sqrt{2}\, dA = \sqrt{2}\pi$

(b) (6 pts.) The portion of the surface S lying on the cylinder $x^2 + y^2 = 1$ may be parametrized by $\mathbf{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z \rangle$. Find the bounds for θ and z and then calculate the surface area of that portion of S.

$0 \leq \theta \leq 2\pi \qquad 0 \leq z \leq 2 - y \quad$ so $\quad 0 \leq z \leq 2 - \sin\theta$

$r_\theta = \langle -\sin\theta, \cos\theta, 1 \rangle \qquad r_z = \langle 0, 0, 1 \rangle$

$r_\theta \times r_z = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 1 \\ 0 & 0 & 1 \end{vmatrix} = i\cos\theta - j(-\sin\theta) + 0k = \langle \cos\theta, \sin\theta, 0 \rangle$

$\Rightarrow |r_\theta \times r_z| = 1 \quad \& \quad S_2 = \int_0^{2\pi} \int_0^{2-\sin\theta} 1\, dz\, d\theta = \int_0^{2\pi} 2 - \sin\theta\, d\theta$

$= 4\pi - \left(\cos\theta \Big|_0^{2\pi}\right)$

$= 4\pi$

(c) (4 pts.) What is the total surface area of S?

S_3 : disk ~~circle~~ of radius 1 : so π

$S = S_1 + S_2 + S_3 = \sqrt{2}\pi + 4\pi + \pi = (5 + \sqrt{2})\pi$