1. (10 points) Short answer questions.

   (a) (3 pts) True/False (choose one, you do not have to justify it) The following parametrizes a cylinder $\{x^2 + y^2 = 4\}$

$$x = 2\cos\theta, \quad y = 2\sin\theta, \quad z = z.$$

$$x^2 + y^2 = 4\cos^2\theta + 4\sin^2\theta = 4$$

   (b) (3 pts) Let $f(x, y, z) = z^2 - x^2 - y^2$. Which type of quadric surface is the level surface of f at $k = 0$?

$$z^2 - x^2 - y^2 = k = 0$$

$$z^2 = x^2 + y^2$$

$$\boxed{\text{cone}}$$

   (c) (4 pts) Complete the sentence (you do not have to justify it): Let $z = f(x, y)$, then the maximum rate of change of f is given in the direction of $\underline{\nabla f}$, and the rate of change is $\underline{|\nabla f|}$ (the magnitude of the gradient vector)

2. (12 points) (a) Find the length of the curve $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle$ from $t = 0$ to $t = 1$.

$$\vec{r}\,'(t) = \langle 2, 2t, t^2 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2} = 2 + t^2$$

$$\int_0^1 |\vec{r}\,'(t)|\,dt = \int_0^1 2 + t^2\,dt = 2t + \frac{1}{3}t^3 \Big|_0^1 = 2\frac{1}{3} = \frac{7}{3}$$

(b) Now, let $\mathbf{r}(t)$ be any other curve that you know is parametrized with respect to arc length. What is the length of $\mathbf{r}(t)$ from $t = a$ to $t = b$, $0 \le a < b$? (The curve is parametrized with respect to the arc length starting from $t = 0$ in the direction of increasing t.)

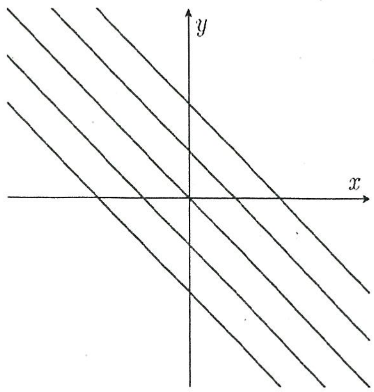
Since $\vec{r}(t)$ is parametrized with respect to arclength, then we know $|\vec{r}\,'(t)| = 1$.

Therefore,

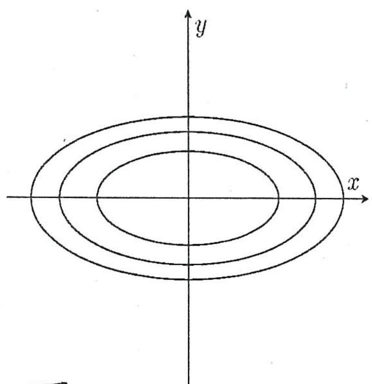
$$\int_a^b |\vec{r}\,'(t)|\,dt = \int_a^b 1\,dt = t\Big|_a^b = b - a$$

3 (~~8~~ 9 points) Match each contour map with a function from the choices on the right.

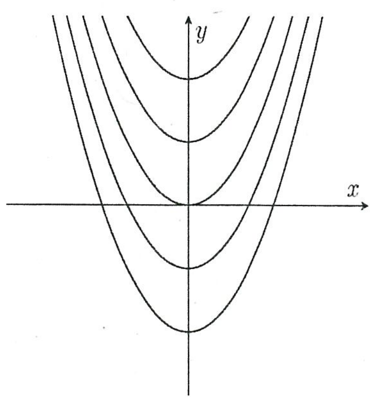
1. D



2. B



3. F



A. $f(x, y) = x - y$

$x - y = k$

$x - k = y$

B. $f(x, y) = \frac{x^2}{4} + y^2$

$\frac{x^2}{4} + y^2 = k$

C. $f(x, y) = x - y^2$

D. $f(x, y) = x + y$

$x + y = k$

$y = k - x$

E. $f(x, y) = x^2 + y^2$

F. $f(x, y) = y - x^2$

$y - x^2 = k$

$y = x^2 + k$

4. (12 points) Demonstrate that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

Let $\gamma_1 : \mathbb{R} \longrightarrow \mathbb{R}^2$

$t \longmapsto (t, t^2)$

and let

$\gamma_2 : \mathbb{R} \longrightarrow \mathbb{R}^2$

$t \longmapsto (t, -t^2)$

then on γ_1 the limit becomes

$$\lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2}$$

and on γ_2 the limit becomes

$$\lim_{t\to 0} \frac{-t^4}{2t^4} = -\frac{1}{2}$$

Since $\frac{1}{2} \neq -\frac{1}{2}$ the limit D.N.E.

5. (10 points) Find an equation of the tangent plane to the hyperbolic paraboloid

Solution 1

$$z = x^2 - y^2 \text{ at the point } P(1,1,0).$$

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f(x,y) = x^2 - y^2$$

$$f_x = 2x\Big|_{(1,1,0)} = 2$$

$$f_y = -2y\Big|_{(1,1,0)} = -2$$

$$z - 0 = 2(x-1) - 2(y-1)$$

Solution 2

$$F(x,y,z) = x^2 - y^2 - z = 0$$

$$\nabla F\Big|_{(1,1,0)} = \langle 2x, -2y, -1\rangle\Big|_{(1,1,0)} = \langle 2,-2,-1\rangle = \vec{n}$$

Plane

$$\vec{n} \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$$

$$\langle 2,-2,-1\rangle \cdot \langle x-1, y-1, z\rangle = 0$$

$$2(x-1) - 2(y-1) - z = 0$$

6. (12 points) Suppose $w = f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$.

(a) Write out the chain rule for w. That is, find expressions for $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$. You do not need to show any work.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

(b) Using your answer from part (a), find an expression for $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right)$.

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right)\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y}\right)\cdot\frac{\partial y}{\partial t}$$

$$+ \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial t}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial t}$$

$$\boxed{\frac{\partial^2 w}{\partial t^2} = \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 w}{\partial y \partial x}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial t^2} + \left(\frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial t} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial x}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t^2}}$$

7. (11 points) Let

$$f(x, y) = e^x \sin(y).$$

(a) Find the gradient of f at any point (x, y).

$\nabla f(x,y) = \langle e^x \sin(y), e^x \cos(y) \rangle$

(b) Find the gradient of f at the point $(1, 0)$.

$\nabla f(1,0) = \langle 0, e \rangle = e\vec{j}$

(c) Find the rate of change of f at the point $(1, 0)$ in the direction of the vector

$$\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}.$$

Normalize $\vec{v}$ $\qquad \|\vec{v}\| = \sqrt{9+16} = \sqrt{25} = 5$

$\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$

Recall

$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$

So

$D_{\vec{u}} f(1,0) = \langle 0, e \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{4e}{5}$

8. (12 points) Let $f(x, y) = 10 - x^2 - y^2 - x$.

(a) Find all the **local** maxima and minima of $f(x, y)$, and saddle points if any. Give both location and value.

$f_x = -2x-1$, $f_y = -2y$

$f_x = f_y = 0 \iff x = \frac{-1}{2}$ & $y = 0$

&

$f_{xx} = -2$, $f_{yy} = -2$ & $f_{xy} = 0$

So $D(x,y) = 4 > 0$ & $f_{xx} < 0$

So at $(\frac{-1}{2}, 0)$ $f(x,y)$ has

a Max Value of $f(\frac{-1}{2}, 0) = \frac{41}{4}$

(b) Restrict f to the closed region $D = \{(x, y) |\ x^2 + y^2 \leq 9\}$. What are the **absolute (global)** maximum and minimum of $f(x, y)$ over D? Give both location and value.

Can re write D as $D = \{(x,y) | -3 \leq x \leq 3,\ y = \pm\sqrt{9-x^2}\}$

Let $g(x) = f(x, y = \pm\sqrt{9-x^2})$

$= 1 - x$

$g(x)$ has no crit pts. So min for $g(x)$ at $x = 3$

& max at $x = -3$.

So $f(-3,0) = 4$ & $f(3,0) = -2$, & from Part (a),

$f(-\frac{1}{2}, 0) = \frac{41}{4}$ so global max is $\frac{41}{4}$ & global

min is -2.

9. (12 points) Use Lagrange multipliers to find the maximum and minimum values of the function

$$f(x, y, z) = 12x - 6z \text{ subject to the constraint } x^2 - 8y^2 + z^2 = 5.$$

$g = x^2 - 8y^2 + z^2$

$\nabla f = \langle 12, 0, -6 \rangle$

$\nabla g = \langle 2x, -16y, 2z \rangle$

$\nabla f = \lambda \cdot \nabla g$

$\langle 12, 0, -6 \rangle = \lambda \langle 2x, -16y, 2z \rangle$

$12 = 2\lambda x$

$0 = -16\lambda y \longrightarrow$ Either $\lambda = 0$ or $y = 0$

$-6 = 2\lambda z$

$x^2 - 8y^2 + z^2 = 5$

if $\lambda = 0 \to$ get $12 = 0$ — impossible

So $\lambda \neq 0 \Rightarrow y = 0.$

$x = \frac{6}{\lambda}$

$z = \frac{-3}{\lambda}$

$5 = x^2 - 8y^2 + z^2 = \left(\frac{6}{\lambda}\right)^2 + 0 + \left(\frac{-3}{\lambda}\right)^2 = \frac{36}{\lambda^2} + \frac{9}{\lambda^2}$

$\Rightarrow \lambda^2 = \frac{45}{5} = 9 \quad \Rightarrow \lambda = \pm 3$

$\lambda = 3$: $x = 2$, $y = 0$, $z = -1$ $\Rightarrow f(2, 0, -1) = 12 \cdot 2 - 6(-1) = 30$

$\lambda = -3$: $x = -2$, $y = 0$, $z = 1$ $\Rightarrow f(-2, 0, 1) = 12(-2) - 6(1) = -30$

Max	Value = 30
Min	Value = -30