University of Colorado Boulder

Math 2400, Midterm 1

Spring 2017

PRINT YOUR NAME: Solutions

PRINT INSTRUCTOR'S NAME: ____________________

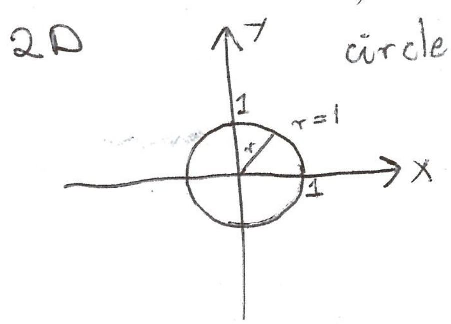
SECTION #: ____________________

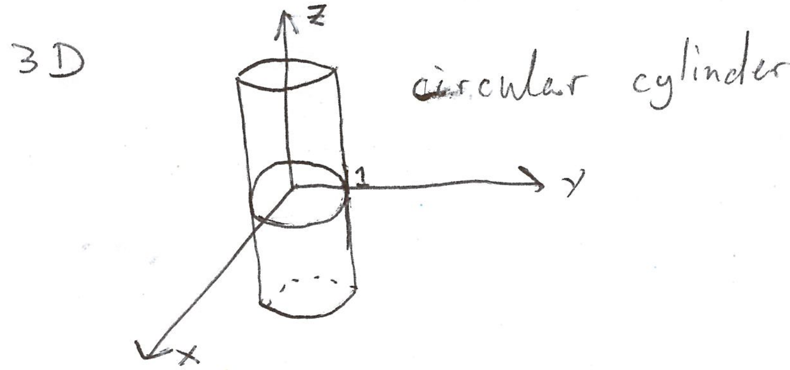
Question	Points	Score
1	14	
2	12	
3	12	
4	12	
5	8	
6	8	
7	12	
8	12	
9	10	
Total:	100	

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit unless indicated otherwise. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like $\ln(3)/2$ as is.
- For multiple choice questions, circle the correct answer.
- Notation: following the book, we use boldface to denote vectors, e.g., $\mathbf{a}, \mathbf{b}$ are vectors.

1. (14 points) Short answer questions

(a) (4 pts, 1+3) Sketch $x^2 + y^2 = 1$ first in 2D (in two dimensions), and then in 3D (in three dimensions).





(b) (3pts) Let $\mathbf{a} = < 1, 2, -1 >$. Normalize $\mathbf{a}$ (i.e., find a vector that points in the same direction as $\mathbf{a}$ but has length 1).

$$|\vec{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$

so $$\frac{\langle 1, 2, -1 \rangle}{\sqrt{6}} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle$$

(c) (correct 3 pts; blank 0 pts, incorrect - 3pts) True/False (choose one, you do not have to justify it). The expression $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$ is meaningful.

(d) (4 pts) Consider the vector-valued function $\mathbf{r}(t) = \langle t, \sqrt{t-4}, \frac{1}{t-4} \rangle$. Find the domain of $\mathbf{r}$.

t: $t \in \mathbb{R}$

$\sqrt{t-4}$: $t - 4 \geq 0$, $t \geq 4$

$\frac{1}{t-4}$: $t - 4 \neq 0$ $t \neq 4$

$\Rightarrow$ domain of $\vec{r}$: $\{ t \in \mathbb{R} : t > 4 \}$

2. (12 points) Let $\mathbf{a} = \langle 1, 0, -1 \rangle$, $\mathbf{b} = \langle 2, -1, -3 \rangle$.

(a) Are $\mathbf{a}$ and $\mathbf{b}$ orthogonal? Justify.

$$\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$$

$$\vec{a} \cdot \vec{b} = 1 \cdot 2 + 0 \cdot (-1) + (-1) \cdot (-3) = 2 + 0 + 3 = 5 \neq 0$$

$\boxed{\text{so } \vec{a} \text{ \& } \vec{b} \text{ are NOT orthogonal}}$

(b) If the vectors are not orthogonal, find the angle between them (you do not have to simplify).

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\sqrt{2}\sqrt{14}}\right)$$

$$\boxed{= \cos^{-1}\left(\frac{5}{\sqrt{2}\sqrt{14}}\right)}$$

$$|\vec{a}| = \sqrt{1^2 + 0 + (-1)^2} = \sqrt{2}$$

$$|\vec{b}| = \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{4+1+9} = \sqrt{14}$$

3. (12 points) Consider the three vectors $\mathbf{u} = \langle 2, 1, 0 \rangle$, $\mathbf{v} = \langle 1, -3, 0 \rangle$ and $\mathbf{w} = \langle 0, 0, 4 \rangle$.

(a) Calculate $\mathbf{u} \times \mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 0 \\ 1 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -3 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} \vec{k}$$

$$= ((1)(0) - (-3)(0))\vec{i} - ((2)(0) - (0)(1))\vec{j} + ((-3)(2) - (1)(1))\vec{k}$$

$$= 0\vec{i} + 0\vec{j} - 7\vec{k}$$

$$= -7\vec{k}$$

(b) What is the *magnitude* of your above answer? What is its geometric meaning?

$|\vec{u} \times \vec{v}| = 7$. This is the area of the parallelogram formed by $\vec{u}$ & $\vec{v}$.

(c) The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form the edges of a parallelepiped. Use your answer in part (b) to find the volume of this parallelepiped.

Notice that $V = A \cdot h$, where $A = |\vec{u} \times \vec{v}|$ &

$h = 4$ (Since $\vec{w}$ is perp. to $\vec{u}$ & $\vec{v}$, the height is simply $|\vec{w}|$. This is not true in general if $\vec{w}$ is not perp. to $\vec{u}$ & $\vec{v}$)

So $V = 28$

Alternatively $V = |\vec{w} \cdot (\vec{u} \times \vec{v})| = |\langle 0,0,4 \rangle \cdot \langle 0,0,-7 \rangle| = |-28| = 28$

(or $V = |\vec{u} \cdot (\vec{w} \times \vec{v})|$ or $|\vec{v} \cdot (\vec{u} \times \vec{w})|$)

4. (12 points) (a) Find the parametric equation of a line that passes through the points $P(6, -2, -1)$ and $Q(3, 0, -4)$. Answers to (a) and (b) may vary. Common solutions given.

$\vec{v} = \vec{PQ} = \vec{Q} - \vec{P} = \langle -3, 2, -3 \rangle$.

Solution 1: $\vec{r}(t) = \langle 6, -2, -1 \rangle + \langle -3, 2, -3 \rangle t$

$= \langle 6 - 3t, -2 + 2t, -1 - 3t \rangle$

Solution 2: $\vec{r}(t) = \langle 3, 0, -4 \rangle + \langle -3, 2, -3 \rangle t$

$= \langle 3 - 3t, 2t, -4 - 3t \rangle$

(b) What are the symmetric equations for the given line?

Solution 1:

$$\frac{x-6}{-3} = \frac{y+2}{2} = \frac{z+1}{-3}$$

Solution 2:

$$\frac{x-3}{-3} = \frac{y}{2} = \frac{z+4}{-3}$$

(c) At what point does this line intersect the yz-plane? yz-plane: $x = 0$

Solution 1:

$6 - 3t = 0$

$t = 2$

$y = -2 + 2(2) = 2$

$z = -1 - 3(2) = -7$

$(0, 2, -7)$

Solution 2:

$3 - 3t = 0$

$t = 1$

$y = 2(1) = 2$

$z = -4 - 3(1) = -7$

$(0, 2, -7)$

5. (8 points) (2 pts each) Given the quadric surface $16x = 4y^2 + z^2$ find the specific traces and identify the surface (you do not need to justify).

(a) For $x = k$ identify the trace.

$16k = 4y^2 + z^2$

A. circle

B. ellipse

C. hyperbola

D. parabola

(b) For $y = k$ identify the trace.

$16x = 4k^2 + z^2$

A. circle

B. ellipse

C. hyperbola

D. parabola

(c) For $z = k$ identify the trace.

$16x = 4y^2 + k^2$

A. circle

B. ellipse

C. hyperbola

D. parabola

(d) Describe the quadric surface.

A. ellipsoid

B. elliptic paraboloid

C. hyperbolic paraboloid

D. cone

E. hyperboloid of one sheet

F. hyperboloid of two sheets

6. (8 points) In this question, you do not need to justify your answer.

(a) Describe in words or sketch the curve of the intersection of

$$C = \{\, (r, \theta, z) \,|\, r = \frac{1}{\sqrt{2}},\ z \geq 0\,\}$$

with

$$D = \{\, (r, \theta, z) \,|\, 2r^2 + z^2 = 1\,\}.$$

Either try to picture it or plug in.

If plug in, then:

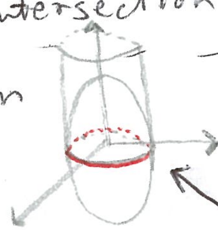
$r = \frac{1}{\sqrt{2}}$ from C into D: $2\left(\frac{1}{\sqrt{2}}\right)^2 + z^2 = 1$

$1 + z^2 = 1$

$z^2 = 0$

so intersection when $z = 0$ & $r = \frac{1}{\sqrt{2}}$ $\Rightarrow$ $\boxed{\text{circle of radius } \frac{1}{\sqrt{2}}}$.

If picturing it, then



red curve:
curve of intersection: circle

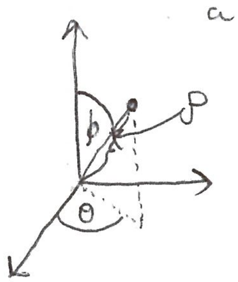
(b) Determine, and then either sketch or describe in words, the type of the solid given by the following inequalities

$$0 \leq \rho \leq 4,\ 0 \leq \phi \leq \frac{\pi}{2}.$$

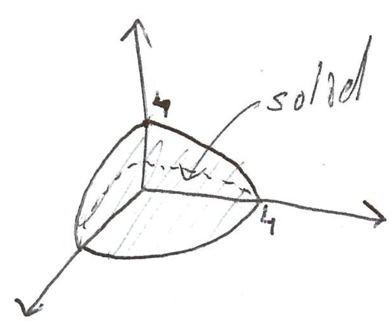
ρ anything from 0 to 4 & ϕ anything from 0 to $\frac{\pi}{2}$,

so a solid sphere: top part, of radius 4, i.e.

a $\boxed{\text{solid top hemisphere of radius 4}}$.



sketch:



7. (12 points) Let $\mathbf{r}(t) = \langle 2\cos t,\ 3\ , \sin t\rangle$.

(a) Find $\mathbf{r}'(t)$.

$$\vec{r}'(t) = \langle -2\sin t,\ 0,\ \cos t \rangle$$

(b) Find $\int_0^{\pi} \mathbf{r}(t)dt$.

$$\int_0^{\pi} \vec{r}(t)\,dt = \left\langle 2\int_0^{\pi} \cos t\,dt,\ \int_0^{\pi} 3\,dt,\ \int_0^{\pi} \sin t\,dt \right\rangle$$

$$= \left\langle 2\sin t\Big|_0^{\pi},\ 3\cdot\pi,\ -\cos t\Big|_0^{\pi} \right\rangle$$

$$= \langle 0,\ 3\pi,\ -(\cos\pi - \cos 0) \rangle$$

$$= \langle 0,\ 3\pi,\ -(-1-1) \rangle$$

$$\boxed{= \langle 0,\ 3\pi,\ 2 \rangle}$$

8. (12 points) Suppose two particles have positions with respect to time given by the vector functions

$$\mathbf{r}_1(t) = 2t\mathbf{i} + \left(t^2 - 6\right)\mathbf{j} + \left(-\frac{1}{3}t^3\right)\mathbf{k}$$

$$\mathbf{r}_2(s) = (2 + 2s)\mathbf{i} + (5 - s)\,\mathbf{j} + (1 - 5s)\,\mathbf{k}$$

for $t \geq 0$ and $s \geq 0$.

(a) Show that the space curves given by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ intersect at a common point.

Need $\vec{r}_1(t) = \vec{r}_2(s)$ for some value of s & t.

so need ① $2t = 2+2s$ ② $t^2 - 6 = 5 - s$ ③ $-\frac{1}{3}t^3 = 1 - 5s$

solving ① for s & pluggi̇ng into ② we get: $t = 1 + s$, $s = t - 1$ (*)

$t^2 - 6 = 5 - (t-1)$

$t^2 - 6 = 5 - t + 1$

$t^2 + t - 12 = 0$

$(t+4)(t-3) = 0$

$t = \cancel{-4}$, $\boxed{t = 3}$

(Note, we're given $t \geq 0$)

Plug in $t=3$ into (*): $\boxed{s = 3 - 1 = 2}$

(Check in 3rd eqn: $-\frac{1}{3}3^3 = 1 - 5(2)$

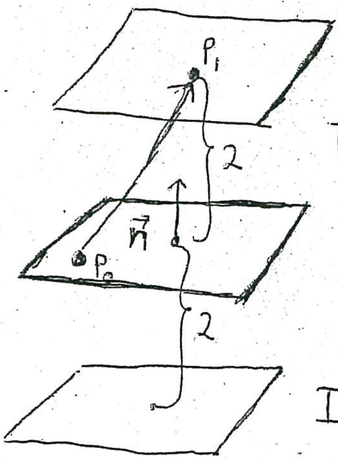
$-\frac{1}{3}27 = 1 - 10$ ✓)

So Yes, the curves intersect at $(2+4, 5-2, 1-10)$
$= (6, 3, -9)$

(b) If $t = s$, will the two particles ever collide? Justify your answer.

The space curves intersect at $(6, 3, -9)$. In order for the collision to occur we need $\vec{r}_1(t) = \vec{r}_2(t)$, but from part a) $\vec{r}_1$ goes through $(6, 3, -9)$ at $t = 3$, and $\vec{r}_2$ at $t = 2$, so no, the particles do NOT collide.

9. (10 points) Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.



① Using the distance between planes formula

The planes must have parallel normal vectors so we can assume the plane is of the form

$$X + 2y - 2z = d$$

If $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ then the distance between the planes is given by

$$2 = D = \frac{|\vec{n} \cdot \overrightarrow{P_0 P_1}|}{|\vec{n}|} = \frac{|(x_1 + 2y_1 + -2z_1) - (x_0 + 2y_0 - 2z_0)|}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{|d-1|}{3}$$

$$\Leftrightarrow \quad 6 = |d-1| \quad \Leftrightarrow \quad d = 7, -5$$

$$\boxed{X + 2y - 2z = 7 \quad \text{and} \quad X + 2y - 2z = -5}$$

② Pick a point P on the plane and travel 2 units along the unit normal vector to get a point on the desired plane

$$P + 2\frac{\vec{n}}{|\vec{n}|} = (1,0,0) + \frac{2}{3}(1,2,-2) = \left(\frac{5}{3}, \frac{4}{3}, -\frac{4}{3}\right)$$

$$P - 2\frac{\vec{n}}{|\vec{n}|} = (1,0,0) - \frac{2}{3}(1,2,-2) = \left(\frac{1}{3}, -\frac{4}{3}, \frac{4}{3}\right)$$

Plugging these points into the equations of a plane:

$$\left(X - \frac{5}{3}\right) + 2\left(y - \frac{4}{3}\right) + -2\left(z - \frac{4}{3}\right) = 0 \qquad \left(X - \frac{1}{3}\right) + 2\left(y + \frac{4}{3}\right) - 2\left(z - \frac{4}{3}\right) = 0$$

$$\boxed{X + 2y - 2z = 7} \qquad \boxed{X + 2y - 2z = -5}$$