Math 4001-5001: HW8

Due Friday, 11/01/2019

Problem 8.1 Let f be a continuous function on $[0, 1]$. Prove whether or not the following limit exists. If it exists, find the limit.

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)dx.$$

Problem 8.2 Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Show if f is differentiable at $x_0 \in \mathbb{R}^n$, then f is continuous at x_0.

Problem 8.3 p. 239 # 6.

Problem 8.4 The purpose of this problem is to make a connection between the Chain Rule in Calculus III and the Chain Rule in our class. All functions are assumed to be differentiable.
In the case of $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $F(x) = g(f(x))$, from Calculus I we have, $F'(x) = g'(f(x))f'(x)$, and this agrees exactly with the Chain Rule in Theorem 9.15, when for fixed x, the 1D derivative is thought of as a linear transformation from $\mathbb{R}$ to $\mathbb{R}$. Now consider higher dimensional cases (Hint: use Theorem 9.17):

a. $r : \mathbb{R} \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$, so $r(t) = (x(t), y(t))$ and $f(x, y) \in \mathbb{R}$. Now let

$$F(t) = (f \circ r)(t).$$

From Calculus III, we have

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{1}$$

Let f' and r' be the total derivatives of f and r, respectively, and show (1) is equivalent to $f'(r(t))r'(t)$.

b. $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}$, so $f(x, y) = (r(x, y), \theta(x, y))$ and $g(r, \theta) \in \mathbb{R}$. Now let

$$u(x, y) = (g \circ f)(x, y) = g(r(x, y), \theta(x, y))$$

From Calculus III, we have (finish the statements):

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial r} \cdots. \tag{2}$$

$$\frac{\partial u}{\partial y} = \tag{3}$$

Let f' and g' be the total derivatives of f and g, respectively, and show (2)-(3) are equivalent to $g'(f(x, y))f'(x, y)$.

Extra Problem

Let X, Y be two normed vector spaces, and $A \in L(X, Y)$. Let $||A|| = \sup_{x\neq 0} \frac{||Ax||_Y}{||x||_X}$.

a) Let $||A|| < \infty$. Show

$$||A|| = \sup_{||x||_X \leq 1} ||Ax||_Y = \sup_{||x||_X = 1} ||Ax||_Y = \inf \lambda,$$

where λ satisfies $||Ax||_Y \leq \lambda ||x||_X$.

b) Show if $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $||A|| < \infty$.

c) Let $B(X, Y) = \{A \in L(X, Y) : ||A|| < \infty\}$. Show $||\cdot||$ is a norm on $B(X, Y)$.