Math 4001-5001: HW6

Due Friday, 10/11/2019

Problem 6.1
Let

$$f_n(x) = \frac{e^{x^2}\sinh^2(x)}{(1+e^{x^2}\sinh^2 x)^n}, \quad n = 0, 1, 2, \ldots,$$

and

$$f(x) = \sum_{n=0}^{\infty} f_n(x).$$

a) What is the domain of each f_n? Is each f_n continuous on its domain?

b) Is f well-defined, i.e., what is the domain of f? Justify.

c) Is f continuous on its domain? Justify.

Problem 6.2
Let

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t}\sin(n\pi x), \quad t \geq 0,\ x \in [0,1],$$

where $A_n \in \mathbb{R}, n = 1, 2, 3, \ldots$, and $|A_n| \leq M$ for some finite positive constant M. Show u is well defined on D_0 by showing the series converges uniformly on D_0, where

$$D_0 = \{(x,t) : 0 \leq x \leq 1,\ t > 0\}.$$

Problem 6.3 Prove the corollary on page 152.

Problem 6.4

Let X be a metric space. Show X is compact if and only if every sequence contains a convergent subsequence.
Hint for ($\Rightarrow$): Argue by contradiction. If there was a sequence with no convergent subsequence, use that sequence to construct an open cover of X, such that every set in the cover contains only a finite number of elements of the sequence. Then use compactness to get a contradiction.
Hint for ($\Leftarrow$): Let U_α be an open cover. Consider a *minimal* subcover of U_α. Minimal means that no U_α may be removed from the subcover, if it is to remain a cover of X. Use that every sequence contains a convergent subsequence to argue by contradiction that the minimal subcover must be finite.