Math 4001-5001: HW10

Due Friday, 11/15/2019

Problem 10.1 Let $f : \mathbb{R} \to \mathbb{R}$.

a. If $f \in C^1(\mathbb{R})$ and $f'(x) \neq 1$ for every x, show that f has at most one fixed point.

b. Let $f(x) = x + (1 + e^x)^{-1}$. Show f has no fixed point even though $0 < f'(x) < 1$ for all x (you need to show the lack of the fixed point and the property of the derivative).

c. Now suppose there is $M < 1$ such that $|f'(x)| \leq M$ for all x. Show there exists a unique fixed point.

Problem 10.2 Let $E, F \subset \mathbb{R}^n$ such that E, F are open. Suppose f is a bijection from E onto F, $f \in C^1(E)$, and $f^{-1} \in C^1(F)$ (i.e., f is a diffeomorphism from E onto F). Prove that at each $x \in E$, $f'(x)$ is an isomorphism of $\mathbb{R}^n$.

Problem 10.3 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(r, \theta) = (x(r, \theta), y(r, \theta)) = (r \cos\theta, r \sin\theta)$.

a. Compute the Jacobian matrix of f, i.e., $(\partial_j f_i)$.

b. Use a theorem from class to say at which points f is differentiable and why.

c. Compute the determinant of the Jacobian matrix of f (also simply called the Jacobian of f).

d. Use a theorem from class to find (*familiar*) subsets of $\mathbb{R}^2$, E, F such that f is a diffeomorphism of E onto F.

Problem 10.4 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (e^y \cos x, e^y \sin x)$. Show $\det f'(x, y) \neq 0$ for all (x, y), but f is not $1 - 1$. Why does this not contradict the Inverse Function Theorem?