PROPOSITION 1.27. *Suppose (X,d) is a metric space. Then X is compact if and only if every sequence $\{x_n\}_{n=1}^{\infty} \subset X$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges in X.*

PROOF. Suppose X is compact, but that there is a sequence with no convergent subsequence. For each n, let

$$\delta_n = \inf_{m \neq n} d(x_n, x_m) .$$

If, for some n, $\delta_n = 0$, then there are x_{m_k} such that

$$d(x_n, x_{m_k}) < \frac{1}{k} ,$$

that is, $x_{m_k} \to x_n$ as $k \to \infty$, a contradiction. So $\delta_n > 0 \ \forall\ n$, and

$$\left\{B_{\delta_n}(x_n)\right\}_{n=1}^{\infty} \cup \left(\overline{\bigcup_{n=1}^{\infty} \overline{B_{\delta_n/2}(x_n)}}\right)^c$$

is an open cover of X with no finite subcover, contradicting the compactness of X and establishing the forward implication.

Suppose now that every sequence in X has a convergent subsequence. Let $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ be a minimal open cover of X. By this we mean that no U_α may be removed from the collection if it is to remain a cover of X. Thus for each $\alpha \in \mathcal{I}$, $\exists\ x_\alpha \in X$ such that $x_\alpha \in U_\alpha$ but $x_\alpha \notin U_\beta$ $\forall\, \beta \neq \alpha$. If $\mathcal{I}$ is infinite, we can choose $\alpha_n \in \mathcal{I}$ for $n = 1, 2, \ldots$ and a subsequence that converges:

$$x_{\alpha_{n_k}} \to x \in X \quad \text{as} \quad k \to \infty .$$

Now $x \in U_\gamma$ for some $\gamma \in \mathcal{I}$. But then $\exists\ N > 0$ such that for all $k \geq N$, $x_{\alpha_{n_k}} \in U_\gamma$, a contradiction. Thus any minimal open cover is finite, and so X is compact. ☐