University of Colorado Boulder

Math 5001, Midterm, Part 2, Take-Home

& Math 4001 Fall 2019

NAME: Solutions for both grad & undergrad class.

Question	Points	Score
1	20	
2	20	
Total:	40	

- Only the course textbook by Rudin, lecture notes and your HW solutions are allowed.
- Read instructions carefully. Show all your reasoning and work for full credit unless indicated otherwise.
- On my honor, as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance.
- You can type or handwrite your answers. Copy the statement of each problem.

1. (20 points) If $\sum_n a_n$ and $\sum_n b_n$ are two series of nonnegative real numbers, prove that $\sum_n a_n$ and $\sum_n b_n$ converge if and only if $\sum_n \sqrt{a_n^2 + b_n^2}$ converges.

2. (20 points) (a) (15pts) Prove or disprove: If $\{f_n\}$ is a sequence of continuous functions converging uniformly on $[0,1]$ to f, then

$$\lim_{n\to\infty} \int_0^{1-\frac{1}{n}} f_n(x)dx = \int_0^1 f(x)dx.$$

   (b) (5pts) Prove or disprove: If each $f_n \geq 0$ on $[0,1]$ and $\{f_n\}$ is a sequence of continuous functions such that $\sum_n f_n$ converge uniformly on $[0,1]$ to f, then

$$\sum_{n=1}^{\infty} \int_0^{1-\frac{1}{n}} f_n(x)dx = \int_0^1 f(x)dx.$$

1. By HW Problem 1.1 c), we have $\exists$ constants $c, d > 0$ such that if $x = (a, b)$ then

$$c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$$

$$\Leftrightarrow \quad c(a+b) \leq \sqrt{a^2+b^2} \leq d(a+b) \qquad (*)$$

(Can also reprove this directly now: for this problem we need

$$0 \leq a, b \leq \sqrt{a^2+b^2} \leq a+b \qquad (**)$$

By given $0 \leq a, b$. Also $a = |a| = \sqrt{a^2} \leq \sqrt{a^2+b^2}$ since $b^2 \geq 0$.

Also $\sqrt{a^2+b^2} \leq a+b \Leftrightarrow a^2+b^2 \leq (a+b)^2$ — x^2 & $\sqrt{x}$ are increasing functions for $x \geq 0$.

Now suppose $\sum_n a_n$ & $\sum_n b_n$ converge.

We could use Thm 3.24, but we will just use the comparison test.

First $\sum a_n + b_n$ converges (do a direct proof or quote Thm 3.47)

Then $|\sqrt{a_n+b_n}| = \sqrt{a_n+b_n} \leq a_n + b_n \ \forall n$ by $(**)$.

$\Rightarrow$ The series $\sum \sqrt{a_n+b_n}$ converges by the comparison test.

Now suppose $\sum_n \sqrt{a_n^2+b_n^2}$ converges.

By (**) $|a_n| = a_n \leq \sqrt{a_n^2+b_n^2}$.

So $\sum a_n$ converges by the comparison test. Similarly, the same holds for $\sum b_n$ since

$$|b_n| \leq \sqrt{a_n^2+b_n^2}.$$

$\square$

2 a)

[redacted] [redacted] Prove or disprove: If $\{f_n\}$ is a sequence of continuous functions converging uniformly on $[0,1]$ to f, then

$$\lim_{n\to\infty}\int_0^{1-\frac{1}{n}} f_n(x)dx = \int_0^1 f(x)dx.$$

Proof: Let $\varepsilon > 0$.

Since $\{f_n\}$ are continuous on $[0,1]$, f is cont. and $\|f\|_\infty \le M$.

So $\int_0^1 f$ is well defined.

Method 1:

Let N_1 be an integer so that $\frac{1}{N_1} \le \frac{\varepsilon}{M}$.

And N_2 s.t. $\|f_n - f\|_\infty < \varepsilon$ if $n \ge N_2$.

Let $N = \max\{N_1, N_2\}$.

Consider:

$$\begin{aligned}\left|\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f\right| &\le \left|\int_0^{1-\frac{1}{n}} (f_n - f)\right| + \left|\int_0^{1-\frac{1}{n}} f - \int_0^1 f\right| \\ &\le \|f_n - f\|_\infty \left(1-\tfrac{1}{n}\right) + \left|\int_{1-\frac{1}{n}}^1 f\right| \\ &\le \|f_n - f\|_\infty + \|f\|_\infty \tfrac{1}{n} \le \varepsilon + M\tfrac{1}{n} \\ &\le \varepsilon + \varepsilon = 2\varepsilon \quad \text{if } n \ge N.\end{aligned}$$

$\square$

Method 2: f_n are continuous on $[0,1]$ $\Rightarrow$ each f_n is bounded, $\|f_n\| \le M_n$.

$f_n \to f$ uniformly $\Rightarrow$ f_n are uniformly bounded: $\|f_n\|_\infty \le M$.

Hence

$$\left|\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f\right| \le \underbrace{\left|\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f_n\right|}_{\le \left|\int_{1-\frac{1}{n}}^1 f_n\right|} + \underbrace{\left|\int_0^1 f_n - \int_0^1 f\right|}_{\longrightarrow 0}$$

since $f_n \in \mathcal{R}$ and $f_n \to f$ unifor (f_n are cont.)

$$\le M\tfrac{1}{n} \longrightarrow 0 \text{ as } n \to \infty$$

$\square$

(b) (5pts) Prove or disprove: If each $f_n \geq 0$ on $[0,1]$ and $\{f_n\}$ is a sequence of continuous functions such that $\sum_n f_n$ converge uniformly on $[0,1]$ to f, then

$$\sum_{n=1}^{\infty} \int_0^{1-\frac{1}{n}} f_n(x)dx = \int_0^1 f(x)dx.$$

Proof: The statement is false.

Consider $f_n(x) = \frac{1}{2^n}\chi_{[0,1]}$ then $\sum f_n(x)$ converges uniformly to $f(x) = \chi_{[0,1]}$ by the geometric series.

We then have $\sum_{n=1}^{\infty} \int_0^1 \frac{1}{2^n}dx = \int_0^1 \frac{1}{2}dx + \sum_{n=2}^{\infty}\int_0^1 \frac{1}{2^n}dx = \int_0^1 1\,dx = 1$ by HW/ Corollary p.152.

Now consider

$$\sum_{n=1}^{\infty} \int_0^{1-\frac{1}{n}} f_n(x)\,dx = \int_0^{1-1} \frac{1}{2}dx\ dx + \sum_{n=2}^{\infty} \int_0^{1-\frac{1}{n}} \frac{1}{2^n}\,dx$$

$$= \sum_{n=2}^{\infty} \int_0^{1-\frac{1}{n}} \frac{1}{2^n}\,dx \neq 1$$

b/c $\int_0^1 \frac{1}{2}dx = \frac{1}{2}$ & $\int_0^{1-1} \frac{1}{2}\,dx = 0$