University of Colorado Boulder

Math 5001, Midterm, Part 1, In-Class

Fall 2019

NAME: Solutions

Question	Points	Score
1	20	
2	20	
3	20	
Total:	60	

- Read instructions carefully. Show all your reasoning and work for full credit unless indicated otherwise.

1. (20 points) Short answer questions.

   (a) (2 pts) [2 pts correct; -2 pts incorrect, 1 pts blank] True or False (no work is needed): Let k be a fixed positive integer, then $\mathbb{R}^k$ is complete.

   (b) (10 pts) Finish the definition and then write the Cauchy criterion for convergence: A series Σa_n converges if and only if
   the sequence of partial sums converges.

   Cauchy criterion for convergence: Let $\epsilon > 0$, then the series Σa_n converges if
   $\exists N$ s.t. if $m \geq n \geq N$, then

   $$\left| \sum_{i=n}^{m} a_i \right| < \varepsilon.$$

   (c) (4 pts) True or False (no work is needed): Given the alernating harmonic series

   $$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$

   there exists a rearrangment of the series such that it converges to $\tan(\sinh(\cos(\pi e)))$.

   (d) (4 pts) Finish the statement of the Arzela-Ascoli Theorem: Let K be a compact metric space. A set A in $C(K)$ is compact if and only if
   A is closed, bounded & equicontinuous.

2. (20 points) (a) Prove or disprove: the following series converges (you can quote any theorem from class):

$$\sum_{n=1}^{\infty} \frac{1}{n + n^{\frac{3}{2}}}$$

The series converges by the comparison test w/ the p series w/ $p = \frac{3}{2}$,

b/c we have

$$\frac{1}{n + n^{3/2}} < \frac{1}{n^{3/2}} \quad (\Leftrightarrow n^{3/2} < n + n^{3/2} \ \forall n \geq 1)$$

(b) Show that if in the ratio test one can take a true limit (not just lim sup) and obtain 1, then the root test will also give 1.

Proof:

This follows from Thm 0.1 in the back & the fact that $\liminf \leq \limsup$. ($\circledast$)

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$

So by (0.1), (0.2) & ($\circledast$), we have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \Rightarrow \limsup_{n\to\infty} \sqrt[n]{a_n} = 1.$$

$\square$

3. (20 points) Consider the following space:

$$C^1([0,1]) = \{f : [0,1] \to \mathbb{R} : \ f \text{ and } f' \text{ are continuous on } \ [0,1]\}.$$

Let

$$||f|| = ||f||_\infty + ||f'||_\infty. \tag{0.1}$$

Using that $||\cdot||_\infty$ defines a norm on $C([0,1])$, one can show (0.1) defines a norm on $C^1([0,1])$.

Show $C^1([0,1])$ is a complete space with this norm. (You can quote any theorem from class.)

Proof: Let $\{f_n\}$ be a Cauchy sequence in C^1.
Then $\forall \varepsilon > 0 \ \exists N$ s.t. if $m, n \geq N$, then

$$||f_n - f_m||_\infty + ||f_n' - f_m'||_\infty < \varepsilon.$$

This means $\{f_n\}$ & $\{f_n'\}$ are Cauchy in $||\cdot||_\infty$.
B/c $C([0,1])$ is a Banach space,

$$f_n \to f \text{ in } ||\cdot||_\infty \text{ for some } f \in C([0,1])$$

and $f_n' \to g$ in $||\cdot||_\infty$ — " — $g \in C([0,1])$.

B/c $f_n \to f$ in $|| \, ||_\infty$, ~~it~~ converges ~~p~~twise $\forall x_0 \in [0,1]$,
so Thm 0.2 (Thm 7.17 in the book) applies and
$g = f'$. And b/c $g \in C([0,1])$, the
sequence converges in $C^1([0,1])$ as needed. $\square$

Theorems that may or may not be useful:

Theorem 0.1. *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf \frac{c_{n+1}}{c_n} < \liminf \sqrt[n]{c_n}, \tag{0.2}$$

$$\limsup \sqrt[n]{c_n} < \limsup \frac{c_{n+1}}{c_n}. \tag{0.3}$$

Theorem 0.2. *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x), \quad (a \leq x \leq b).$$