University of Colorado Boulder

Math 4001, Midterm, Part 1, In-class

Fall 2019

NAME: Solutions

Question	Points	Score
1	20	
2	20	
3	20	
Total:	60	

- No notes, textbooks, and no calculators or any electronic devices are allowed at any time.
- Read instructions carefully. Show all your reasoning and work for full credit unless indicated otherwise.

1. (20 points) Short answer questions.

   (a) (2 pts) [2 pts correct; -2 pts incorrect, 1 pts blank] True or False (no work is needed): Let k be a fixed positive integer, then $\mathbb{R}^k$ is complete.

   (b) (10 pts) Finish the definition and then write the Cauchy criterion for convergence: A series Σa_n converges if and only if
   the sequence of partial sums converges.

   Cauchy criterion for convergence: Let $\epsilon > 0$, then the series Σa_n converges if
   $\exists N$ s.t. if $m \geq n \geq N$, then

   $$\left| \sum_{i=n}^{m} a_i \right| < \varepsilon.$$

   (c) (4 pts) True or False (no work is needed): Given the alernating harmonic series

   $$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$

   there exists a rearrangment of the series such that it converges to $\tan(\sinh(\cos(\pi e)))$.

   (d) (4 pts) Finish the statement of the Arzela-Ascoli Theorem: Let K be a compact metric space. A set A in $C(K)$ is compact if and only if
   A is closed, bounded & equicontinuous.

2. (20 points) (a) Prove or disprove: the following series converges (you can quote any theorem from class):

$$\sum_{n=1}^{\infty} \frac{1}{n + n^{\frac{3}{2}}}$$

The series converges by the comparison test w/ the p series w/ $p = \frac{3}{2}$,

b/c we have

$$\frac{1}{n + n^{3/2}} < \frac{1}{n^{3/2}} \quad (\Leftrightarrow n^{3/2} < n + n^{3/2} \ \forall n \geq 1)$$

(b) Show that if in the ratio test one can take a true limit (not just lim sup) and obtain 1, then the root test will also give 1.

Proof:

This follows from Thm 0.1 in the back & the fact that $\liminf \leq \limsup$. (*)

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1,$$

So by (0.1), (0.2) & (*), we have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \Rightarrow \limsup_{n\to\infty} \sqrt[n]{a_n} = 1.$$

$\square$

3. (20 points) Consider the following space:

$$C^1([0,1]) = \{f : [0,1] \to \mathbb{R} : \ f \text{ and } f' \text{ are continuous on } [0,1]\}.$$

Let

$$||f|| = ||f||_\infty + ||f'||_\infty.$$

Show this defines a norm on $C^1([0,1])$. (You can use that $||\cdot||_\infty$ defines a norm on $C([0,1])$.)

First, we observe $\|f\| \geq 0$ since $\|g\|_\infty \geq 0 \ \forall g : [0,1] \to \mathbb{R}$.
Second, we have $\|f\| < \infty$, b/c f & f' are cont. on $[0,1]$.

Next, we show $\|\lambda f\| = |\lambda| \|f\|$.

$$\|\lambda f\| = \|\lambda f\|_\infty + \|(\lambda f)'\|_\infty = |\lambda| \|f\|_\infty + \|\lambda f'\|_\infty$$
$$= |\lambda| (\|f\|_\infty + \|f'\|_\infty)$$

(b/c $\|\cdot\|_\infty$ is a norm, $\|\lambda g\|_\infty = |\lambda| \|g\|_\infty \ \forall g$.)

To see $\|f\| = 0 \iff f = 0$ on $[0,1]$,
if $f = 0$ on $[0,1]$, then $f' = 0$ on $[0,1]$,
so $\|f\| = \|0\|_\infty + \|0\|_\infty = 0$.
Next if $\|f\| = 0$, we have $\|f\|_\infty = 0$ & $\|f'\|_\infty = 0$.
B/c $\|\cdot\|_\infty$ is a norm $f = 0$ on $[0,1]$
(and $f' = 0$, but we already have $f = 0$),

Finally, for the triangle inequality,

$$\|f+g\| = \|f+g\|_\infty + \|(f+g)'\|_\infty$$
$$= \|f+g\|_\infty + \|f'+g'\|_\infty$$
$$\leq \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty \quad \text{(b/c } \|\cdot\|_\infty \text{ is a norm)}$$
$$= \|f\| + \|g\| \text{ as needed.}$$

Theorems that may or may not be useful:

Theorem 0.1. *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf \frac{c_{n+1}}{c_n} < \liminf \sqrt[n]{c_n}, \tag{0.1}$$

$$\limsup \sqrt[n]{c_n} < \limsup \frac{c_{n+1}}{c_n}. \tag{0.2}$$

Theorem 0.2. *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x), \quad (a \le x \le b).$$