Problem 2.3: If a_n forms a non-increasing sequence and d is a fixed natural number such that $d > 1$, show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} d^k a_{d^k}$ converges.

Suppose $\sum_{n=1}^{\infty} a_n$ is convergent:

$$a_1 + \frac{1}{d}\sum_{n=1}^{N} d^n a_{d^n} = a_1 + \sum_{n=1}^{N} d^{n-1} a_{d^n}$$

$$= a_1 + a_d + \boxed{d a_{d^2}} + \dots + \boxed{d^{N-1} a_{d^N}}$$

uses a_n are non-increasing

start at d^2-d+1 to make sure $d^2-d+1 > d$ (e.g. if $d=2$)

$$\boxed{\le \cancel{a_{d^2-d}} + a_{d^2-d+1} + \dots + a_{d^2}} = \sum_{n=d^2-d+1}^{d^2} a_n$$

Now, there are also exactly d terms (and not $d+1$)

$$\boxed{\le \cancel{a_{d^N-d^{N-1}}} + a_{d^N-d^{N-1}+1} + \dots + a_{d^N}} = \sum_{n=d^N-d^{N-1}+1}^{d^N} a_n$$

$$\le a_1 + a_2 + \sum_{n=d^2-d+1}^{d^2} a_n + \sum_{n=d^3-d^2+1}^{d^3} a_n + \dots + \sum_{n=d^N-d^{N-1}+1}^{d^N} a_n \quad (*)$$

$$\le a_1 + a_2 + \boxed{\sum_{n=3}^{d^2-d-1} a_n} + \sum_{n=d^2-d}^{d^2} a_n + \boxed{\sum_{n=d^2+1}^{d^3-d^2-1} a_n} + \sum_{n=d^3-d^2}^{d^3} a_n + \dots + \sum_{n=d^N-d^{N-1}}^{d^N} a_n$$

Adding in additional positive terms (if anything is missing)

$$= \sum_{n=1}^{d^N} a_n \le \sum_{n=1}^{\infty} a_n$$

In general, can say $(*) \le \sum_{n=1}^{d^N} a_n = S_{d^N} < \infty$.

Therefore $a_1 + \frac{1}{d}\sum_{n=1}^{\infty} d^n a_{d^k}$ is convergent since its partial sums are bounded by $\sum_{n=1}^{\infty} a_n$ which is convergent by hypothesis. But this expression is simply $\sum_{k=1}^{\infty} d^k a_{d^k}$ with an added constant and multiplied by $\frac{1}{d}$ which is a constant so $\sum_{k=0}^{\infty} d^k a_{d^k}$ must also converge. Consequently whenever $\sum_{n=1}^{\infty} a_n$ converges so too does $\sum_{k=0}^{\infty} d^k a_{d^k}$.

Now suppose $\sum_{k=0}^{\infty} d^k a_{d^k}$ is convergent. Because d is a constant the series $d\sum_{k=0}^{\infty} d^k a_{d^k}$ is also convergent:

(Observe, $N < d^N + d^{N-1} + \ldots + d$)

$$\sum_{n=1}^{N} a_n \leq \sum_{n=1}^{d^N+d^{N-1}+\ldots+d} a_n$$

$$= \boxed{\sum_{n=1}^{d} a_n} + \boxed{\sum_{n=d+1}^{d^2+d} a_n} + \boxed{\sum_{n=d^2+d+1}^{d^3+d^2+d+1} a_n} + \ldots + \boxed{\sum_{n=d^{N-1}+d^{N-2}+\ldots+d+1}^{d^N+d^{N-1}+\ldots+d} a_n}$$

$$\leq \boxed{da_1} + \boxed{d^2 a_{d+1}} + \boxed{d^3 a_{d^2+d+1}} + \ldots + \boxed{d^N a_{d^{N-1}+d^{N-2}+\ldots+d+1}}$$

$$\leq \boxed{da_1} + \boxed{d^2 a_d} + \boxed{d^3 a_{d^2}} + \ldots + \boxed{d^N a_{d^{N-1}}} + \boxed{d^{N+1} a_{d^N}}$$

Adding in additional positive terms

$$= \sum_{k=0}^{N} d^{k+1} a_{d^k} \leq d \sum_{k=0}^{\infty} d^k a_{d^k}$$

From this we see that the partial sums $\sum_{n=1}^{N} a_n$ are bounded by $d\sum_{k=0}^{\infty} d^k a_{d^k}$ for any N so whenever $\sum_{k=0}^{\infty} d^k a_{d^k}$ is convergent, so too is $\sum_{n=1}^{\infty} a_n$