Math 3430: Summary for the Exact equations

The purpose of this handout is to give a concise description of the method.

1 The Method

Consider a first order ODE in the form

$$M(x,y) + N(x,y)y' = 0, \tag{1}$$

or

$$M(x,y)dx + N(x,y)dy = 0. \tag{2}$$

If the ODE is exact, then the general solution is given by

$$\Phi(x,y) = c$$

for some function Φ, and where c is an arbitrary constant.

Step 1) Check if the equation is exact. Check if

$$\partial_y M = \partial_x N. \tag{3}$$

If (3) does not hold, stop here. If it does hold, go to Step 2).

Step 2) Here we find the (potential) function $\Phi(x,y)$. There are three ways we can proceed. Because the ODE is exact, we know

$$M = \partial_x \Phi, \quad N = \partial_y \Phi.$$

Method 1) Because $M = \partial_x \Phi$, then

$$\Phi(x,y) = \int M(x,y)dx + g(y). \tag{4}$$

Evaluate the integral involving M, and find g by looking at $\partial_y \Phi$ using (4) and comparing to N.

Method 2) Because $N = \partial_y \Phi$, then

$$\Phi(x,y) = \int N(x,y)dy + h(x) \tag{5}$$

Evaluate the integral involving N, and find h by looking at $\partial_x \Phi$ using (5) and comparing to M.

Method 3) We have

$$\Phi(x,y) = \int M(x,y)dx + g(y) \tag{6}$$

$$\Phi(x,y) = \int N(x,y)dy + h(x) \tag{7}$$

Evaluate both integrals and figure out $g(y)$ and $h(x)$ by inspection.

Step 3) Write the final solution as

$$\Phi(x,y) = c$$

Remark 1.1. *The integrals in equations* (4)-(7) *are indefinite integrals, so when we evaluate them, there will be constants produced, but we absorb these constants into the functions g and h.*