Math 3430

1 Worksheet for the Existence and Uniqueness Theorem

In this worksheet we examine a theorem that tells us when we can have a solution, and when there will be *only one* solution to an initial value problem (IVP).

Theorem 1.1 (Existence and Uniqueness Theorem for the First Order ODEs)**.** *Consider*

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \tag{$\star$}$$

- *(Existence). If $f(x, y)$ is continuous in an open rectangle $R = (a, b) \times (c, d)$ in the xy-plane that contains the point (x_0, y_0), then there exists a solution $y(x)$ to the initial-value problem (IVP) $(\star)$ that is defined in some open interval $I = (\alpha, \beta)$ containing x_0.*
- *(Uniqueness) If the partial derivative $\frac{\partial f}{\partial y}$ is continuous in R, then the solution $y(x)$ of $(\star)$ is unique.*

Remark 1.2. *We can state a STRONGER version of the theorem, which means we could state a theorem with WEAKER assumptions. More precisely, the theorem is still true if the condition on the partial derivative $\frac{\partial f}{\partial y}$ being continuous in R, is replaced by what is called a Lipschitz condition (in the second argument). You are encouraged to look up what this means.*

Exercise 1.3. *This is a nice problem from Analysis:*
Show that if the partial derivative $\frac{\partial f}{\partial y}$ is continuous in R, then f satisfies a Lipschitz condition in R with respect to its second argument.

2 Discussion of the statement of the Theorem

First we take a closer look at what kind of equation the equation $(\star)$ is.

2.1 Format of the equation $(\star)$

We write the ODE again:

- The order of the ODE is:
- The ODE is: (choose one) linear, nonlinear or

Example 1. *Here are some examples of what f we could have on the right hand side:*

There are infinitely many examples.

Remark 2.1. *The theorem is stated for a single equation that is first order. However, it can be used to deduce a theorem for:*

1. *1st order* systems *of ODEs.*
2. *ODEs of higher order.*

We will come back to this in the future lectures.

2.2 Hypothesis and Conclusion

In general, any theorem has two parts: a hypothesis, and a conclusion. If the hypothesis is true, then you get the conclusion.

The theorem above addresses two topics: Existence and Uniqueness of the IVP for an ODE.

2.2.1 Existence statement of the Theorem

The hypothesis of the theorem above is

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The conclusion of the theorem says

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Remark 2.2. *We can rephrase the hypothesis to say that we can find some rectangle* $R = (a, b) \times (c, d)$ *in the xy- plane that contains the point* (x_0, y_0) *such that* f *is continuous in that rectangle. Note, we just need to find one rectangle that works. It does not have to be true for all rectangles.*

This only tells us that there is at least one solution, but it does not tell us if there is ONLY one. For that, we need the second statement in the theorem.

2.2.2 Uniqueness statement of the Theorem

The hypothesis of the theorem above is

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The conclusion of the theorem says

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3 Examples

We now see how we could apply this theorem in practice.

3.1 Example 1

Consider

$$\frac{dy}{dx} = y, \quad y(0) = 1. \tag{1}$$

Can we apply the theorem to say there exists a unique solution to this IVP?

3.2 Example 2

Consider

$$\frac{dy}{dx} = y^{\frac{1}{3}}, \quad y(0) = 0. \tag{2}$$

Can we apply the theorem to say there exists a unique solution to this IVP?

3.3 IMPORTANT

Consider the statement:

If I am in Boulder,

So now, if I am NOT in Boulder, say in Hawaii, then

That is why we have to be careful: if the uniqueness hypothesis of Theorem 1.1 does not hold, it does not mean

It only means that that part of the theorem

Same is true for the existence part of the theorem.

Remark 3.1. *Note, there are theorems that are written as "if and only if" statements (bi-conditionals). In that case, the hypothesis is equivalent to the conclusion. So if the hypothesis is false, so will be the conclusion. Our theorem, Theorem 1.1 is NOT an if and only if statement!*

3.4 Example 3

Consider

$$\frac{dy}{dx} = y^{\frac{1}{3}}, \quad y(0) = 1. \tag{3}$$

Can we apply the theorem to say there exists a unique solution to this IVP?

3.5 Example 4

Consider

$$\frac{dy}{dx} = \frac{\cos x}{\sin y}, \quad y(\pi) = \pi. \tag{4}$$

Can we apply the theorem to say there exists a unique solution to this IVP?