Math 3430, Spring 2019, Systems of equations

We consider systems

$$x' = Ax, \tag{1}$$

where A is a 2×2 matrix with constant real coefficients, and $x : I \to \mathbb{R}^2$. We look for a solution of the form

$$x(t) = ve^{\lambda t},$$

where v is a constant vector and λ is a scalar. Plugging this into (1) gives

$$\lambda v e^{\lambda t} = e^{\lambda t} Av \quad \Leftrightarrow \quad Av = \lambda v. \tag{2}$$

We are interested in finding a general solution, so we do not want to consider vectors $v = 0$. Since by definition the eigenvectors are nonzero, finding vectors v and scalars λ satisfying

$$Av = \lambda v, \tag{3}$$

is equivalent to solving an eigenvalue problem for A. The idea is to find $n = 2$ linearly independent solutions (see Theorem 3 in Section 7.2 and compare to Theorem 2 in Section 2.2). Here is the procedure we follow.
Step 0: Make sure the system is in the matrix form:

$$x' = Ax.$$

Step 1: Find the eigenvalues for the matrix A.
Step 2: Since the characteristic polynomial equation for a 2×2 matrices is quadratic, there are three cases to consider (compare to 2nd order ODE in Chapter 2). We discuss them according to the order of difficulty.
Case 1: Two real distinct eigenvalues: $\lambda_1, \lambda_2 \in \mathbb{R}$. Recalling a theorem from Linear Algebra we know the corresponding eigenvectors will be linearly independent. We find an eigenvector v_1 corresponding to λ_1 and v_2 corresponding to λ_2. The general solution is then given by

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2.$$

Case 2: Complex Eigenvalues: $\alpha \pm i\beta$ ($\alpha, \beta \in \mathbb{R}, \beta \neq 0$). In this case, when we find the eigenvector v_1 corresponding to the eigenvalue $\lambda_1 = \alpha + i\beta$ and the eigenvector v_2 corresponding to $\lambda_2 = \alpha - i\beta$,

$$e^{\lambda_1 t} v_1 \quad \text{and} \quad e^{\lambda_2 t} v_2$$

will be both complex. But since the matrix A is real-valued, it is reasonable to look for real-valued solutions. We follow the same idea as in Chapter 2.

If $w(t)$ is a complex valued solution, we can write it as $v(t) = u_1(t) + iu_2(t)$, where u_1, u_2 are real-valued. Next, because (1) is linear, we can show that both u_1 and u_2 are solutions (exercise!). So we take one eigenvalue $\alpha + i\beta$ and find a corresponding eigenvector v. Next, $ve^{\lambda t}$ is a solution, so we then extract the real and imaginary parts. They can be shown to be

$$u_1(t) = e^{\alpha t}(B_1 \cos \beta t - B_2 \sin \beta t), \quad u_2(t) = e^{\alpha t}(B_2 \cos \beta t + B_1 \sin \beta t),$$

where

$$B_1 = \mathcal{R}e(v), \quad B_2 = \mathcal{I}m(v),$$

where v is an eigenvector associated to the eigenvalue $\alpha + i\beta$. It is another exercise to show u_1 and u_2 are linearly independent. Hence the general solution is given by

$$x(t) = c_1 u_1(t) + c_2 u_2(t).$$

Case 3: Repeated real eigenvalue. This is in some sense the most complicated case. Recall from Linear Algebra

- m_a algebraic multiplicity is the number of times λ appears as a root of $\det(A - \lambda I) = 0$.
- m_g geometric multiplicity is the number of linearly independent eigenvectors corresponding to λ .

Since we have $n = 2$, the repeated eigenvalue means, we have $m_a = 2$. If it turns out that $m_g = 2$ also, then we can just take the linear combination as the solution

$$x(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2.$$

If $m_g = 1$, then we have what is called a *defective eigenvalue.* In this case the general solution can be shown to be given by

$$x(t) = c_1 e^{\lambda t} v + c_2 u_2,$$

where

$$u_2(t) = e^{\lambda t}(tv + w),$$

with v an eigenvector associated to the eigenvalue λ, and w the generalized eigenvector solving $(A - \lambda I)w = v$.