University of Colorado Boulder

Math 3430-002, Exam 2

Fall 2019

NAME: Solutions

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

- No calculators or cell phones or other electronic devices allowed at any time.
- Read instructions carefully. Show all your reasoning and work for full credit unless indicated otherwise.

1. (20 points) Short answer questions.

   (a) Fill in the blanks:

   Consider the ODE

$$ay'' + by' + cy = 0, \ a, b, c \in \mathbb{R}, \ a \neq 0. \tag{0.1}$$

   Suppose y_1 and y_2 are two solutions of (0.1) that are linearly independent on $\mathbb{R}$. Then a general solution y of this equation can be expressed as

   $y(x) = c_1 y_1(x) + c_2 y_2(x)$

   (b) Write the definition of the Wronskian of two functions f and g:

   $W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$

   (c) Fill in the blanks (TWO ways to check if two functions are linearly independent): Two functions f and g are linearly independent on an interval I if their Wronskian

   $W(f, g) \neq 0$ for some $x \in I$, or if

   $f(x) \neq c\,g(x)$ for some constant c.

   (d) Write the general solution of the following ODE

$$y'' + 2y' = 0.$$

$r^2 + 2r = 0$ $\quad r = 0$

$r(r+2) = 0$ $\quad r = -2$

$\boxed{y(x) = c_1 + c_2 e^{-2x}}$

   (e) Write the general solution of the following ODE

$$y'' - 2y' + 5y = 0.$$

$r^2 - 2r + 5 = 0$

$r = \dfrac{2 \pm \sqrt{4 - 4\cdot 5}}{2} = \dfrac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$

$\boxed{y(x) = c_1 e^x \cos 2x + c_2 e^x \sin 2x}$

2. (20 points) Consider the ODE

$$y'' - 2y' + y = f(x).$$

For each of $f(x)$ below write the most educated guess for the trial y_p (**DO NOT solve for the coefficients**):

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0 \quad r = 1$

$y(x) = c_1 e^x + c_2 x e^x$

(a) $f(x) = 4$

$y_p = A$

(b) $f(x) = \sin(x)$ $y_p = A \sin x + B \cos x$

(c) $f(x) = x^2 - 2x$ $y_p = Ax^2 + Bx + C$

(d) $f(x) = e^{+x} + e^{3x}$ $y_p = Ax^2 e^x + Be^{3x}$

3. (20 points) Check directly from the definition if L defined below is a linear or a nonlinear operator. You can use that derivatives of all orders are linear operators, and that multiplication by a function is a linear operator.

$$L = x^2 \frac{d^2}{dx^2} + (\cdot)^2 + 1.$$

Note, this means $Ly = x^2 y'' + y^2 + y$.

Claim: L is nonlinear.

We show $L(cy) \neq cLy$ for any $c \neq 1$ or $c \neq 0$.

$L(cy) = x^2 cy'' + c^2 y^2 + cy$

$cLy = c(x^2 y'' + y^2 + y)$

$c^2 y^2 \neq c y^2$ unless $c = 1$ or $c = 0$.

$\Rightarrow$ L is nonlinear as needed.

4. (20 points) Use the Laplace Transform Table to compute

(a) $\mathcal{L}[2+t^2] =$ $\frac{2}{s} + \frac{2}{s^3}$

(b) $\mathcal{L}[\delta(t-1)] =$ e^{-s}

(c) $\mathcal{L}^{-1}[\frac{s}{s^2+2s+6}] =$ $\mathcal{L}^{-1}\left[\frac{s}{s^2+2s+1-1+6}\right]$

$= \mathcal{L}^{-1}\left[\frac{s}{(s+1)^2+5}\right] = \mathcal{L}^{-1}\left[\frac{s+1-1}{(s+1)^2+5}\right]$

$= \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+\sqrt{5}^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+\sqrt{5}^2}\right]$

$= e^{-t}\cos\sqrt{5}t - \frac{1}{\sqrt{5}}e^{-t}\sin\sqrt{5}t$

5. (20 points) Use the Laplace Transform to solve the following IVP.

$$y' + y = u(t-2), \quad y(0) = 1$$

$$sY - y(0) + Y = e^{-2s}/s$$

$$(s+1)Y = 1 + e^{-2s}/s$$

$$Y = \frac{1}{s+1} + \frac{e^{-2s}}{s(s+1)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$1 = A(s+1) + Bs$

$s = -1$ $\quad s = 0$

$B = -1$ $\quad A = 1$

$$Y = \frac{1}{s+1} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s+1}$$

$$y(t) = e^{-t} + u(t-2) + u(t-2)e^{-(t-2)}$$

$u(t-a)f(t-a)$

$\rightarrow e^{-as}F(s)$

$a = 2$ $\quad F(s) = \frac{1}{s+1}$

$f(t) = e^{-t}$