University of Colorado Boulder

Math 3430, Exam 1

Fall 2019

NAME: Solutions

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

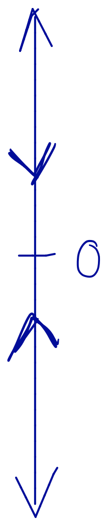
- No calculators or any electronic devices are allowed at any time.
- Read instructions carefully. Show all your reasoning and work for full credit unless indicated otherwise.

1. (20 points) Short answer questions.

   (a) (10pts) Find all equilibrium solutions and determine the stability of each equilibrium.

$$y' = -y.$$

$f(y) = -y \qquad f(y) = 0 \iff y_0 = 0$



$(0, \infty) \qquad \overset{f(y)}{-} \Rightarrow y' < 0$

$(-\infty, 0) \qquad + \Rightarrow y' > 0$

$y_0 = 0$ is a stable sink.

   (b) (10pts) Which type of substitution would allow you to solve the following ODE by transforming it from nonlinear to linear?

$$y^4 y' + y^5 \ln x = e^x$$

Make the substitution, and ~~only~~ fill in the blanks below. DO NOT SOLVE the ODE.

$y' + \ln x \, y = e^x y^{-4} \qquad \alpha = -4$

v= $y^{1-(-4)} = y^5$

(You can choose one of these to compute; use your substitution.)

$y = v^{\frac{1}{5}}$

v'= $5y^4 y'$ or y'= $\frac{1}{5} v^{-4/5} v'$

2. (20 points) (a) (15pts) Find all solutions, i.e., the general solution, to the following ODE. If there is a possibility of dividing by zero, address it.

$$y' = y^2.$$

$\frac{dy}{dx} = y^2$

Case 1: $y \neq 0$

$\frac{dy}{y^2} = dx$

$-\frac{1}{y} = x + C_1$

$y = \frac{1}{C-x}$

Case 2: $y \equiv 0$ Solves the ODE.

General soln: $y = \begin{cases} \frac{1}{C-x} \\ 0 \end{cases}$

$\boxed{\begin{array}{l} \text{Check:} \\ y' = \frac{1}{(C-x)^2} \\ y^2 = \frac{1}{(C-x)^2} \checkmark \end{array}}$

(b) (5pts) Use part a) to solve the following initial value problem. In your final answer, state the interval of existence for the solution y.

$$y' = y^2, \quad y(0) = 1.$$

Use $y = \frac{1}{C-x}$

Then $1 = \frac{1}{C-0}$

$C = 1$

So $y = \frac{1}{1-x}$ $\quad J = (-\infty, 1)$

3. (20 points) Find the general solution of the following ODE. If there is a possibility of dividing by zero, address it.

$$xy' + 3y = 4x\cos(x^4)$$

Standard form: $y' + \frac{3}{x}y = 4\cos x^4$ $\quad x \neq 0$

$I(x) = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = e^{\ln|x|^3} = x^3$

choose $J = (0,\infty)$

$y'x^3 + 3x^2 y = 4x^3\cos x^4$

$\frac{d}{dx}(yx^3) = 4x^3\cos(x^4)$

$yx^3 = \int 4x^3\cos(x^4)\,dx$

$= \sin(x^4) + C$

$$\boxed{y = \frac{\sin(x^4)}{x^3} + \frac{C}{x^3} \quad J = (0,\infty)}$$

check:

$y' = \frac{x^3\cos(x^4)4x^3 - \sin(x^4)3x^2}{x^6} - \frac{3C}{x^4}$

$= \frac{4x^4\cos(x^4) - 3\sin(x^4)}{x^4} - \frac{3C}{x^4}$

$\frac{3}{x}y = \frac{3\sin(x^4)}{x^4} + \frac{3C}{x^4}$ $\Rightarrow y' + \frac{3}{x}y = 4x\cos(x^4)$ ✓

4. (20 points) Find the general solution of the following ODE.

$$(\sin^2 x - 3y^2)y' = -x - 2y \sin x \cos x$$

Check if exact: $M(x,y) = x + 2y \sin x \cos x$

$N(x,y) = \sin^2 x - 3y^2$

Need $\partial_y M = \partial_x N$

$\partial_y M = 2 \sin x \cos x = \partial_x N \Rightarrow$ Exact.

Find $\Phi(x,y) = \int (\sin^2 x - 3y^2)\,dy$

$= (\sin^2 x) y - y^3 + h(x)$

$\partial_x \Phi = M \Rightarrow 2 \sin x \cos x\, y + h'(x)$

$= x + 2y \sin x \cos x$

$h'(x) = x$

$h(x) = \frac{x^2}{2} + C_1$

$\Rightarrow \boxed{\sin^2 x\, y - y^3 + \frac{x^2}{2} = C} \quad (*)$

Check: $\frac{d}{dx}(*)$ gives

$2 \sin x \cos x\, y + \sin^2 x\, y' - 3y^2 y' + x = 0$ ✓

$\int \partial_x \frac{\partial \Phi}{\partial y}\,dy = \frac{\partial \Phi}{\partial x}$

5. (20 points) Recall the following theorem:

Theorem. *Consider*

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{$\star$}$$

- *(Existence). If $f(x,y)$ is continuous in an open rectangle $R = (a,b)\times(c,d)$ in the xy-plane that contains the point (x_0,y_0), then there exists a solution $y(x)$ to the (IVP) $(\star)$ that is defined in an open interval $I = (\alpha,\beta)$ containing x_0.*
- *(Uniqueness) If the partial derivative $\frac{\partial f}{\partial y}$ is continuous in R, then the solution $y(x)$ of $(\star)$ is unique.*

Consider an ODE

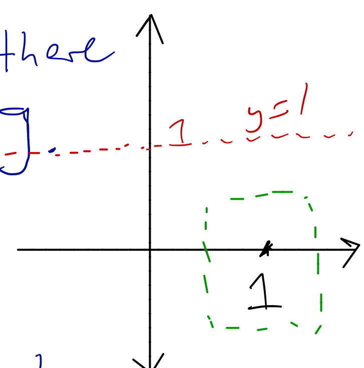
$$\frac{dy}{dx} = e^{x^2}(y-1)^{\frac{2}{3}} \tag{0.1}$$

(a) (18 pts) Given the ODE (0.1) with initial condition

$$y(1) = 0,$$

can we apply the theorem to say there exists a unique solution to this IVP? Justify, and draw your rectangle R.

Existence: $f(x,y) = e^{x^2}(y-1)^{2/3}$ is cont. in $\mathbb{R}^2$ so in any open rectangle R containing $(1,0)$. So yes, we can apply the thm to deduce there exists a soln in some interval J, s.t. $1 \in J$.



Uniqueness: $\partial_y f = \frac{2}{3} e^{x^2} \frac{1}{(y-1)^{\frac{1}{3}}}$ is cont everywhere except $y=1$, but $y_0 = 0$, so we can find a rectangle R s.t. $\partial_y f$ is cont & $(x_0,y_0) \in R$. $\Rightarrow$ the soln is unique.

(b) (2 pts correct, 1 pt blank, -1 pts incorrect). True/False (no work is needed; if there is work, but the reasoning is incorrect, the answer will count as incorrect). Given the ODE (0.1) with initial condition

$$y(1) = 1,$$

the theorem tells us the solution to this IVP is not unique.

$\partial_y f$ is not cont at $(1,1)$ so there can be no rectangle containing $(1,1)$ s.t. $\partial_y f$ is cont. $\Rightarrow$ The Thm does not apply.