Solutions to HW 6

6.1 L is linear if we can show

$$L(\alpha f + \beta g) = \alpha Lf + \beta Lg \quad \forall \alpha, \beta \in \mathbb{R}$$
$\& \ \forall f, g \in C^{k}(J)$,

We can use derivatives & multiplication by a fixed function is a linear operator. 1

We also use that if L_1 & L_2 are two operators, 2
then $(L_1 + L_2)(f) := L_1 f + L_2 f$

6.1 a) $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad a, b, c \in \mathbb{R}$

consider

$$L(\alpha f + \beta g) = \left(a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right)(\alpha f + \beta g)$$

$$= a\frac{d^2}{dx^2}(\alpha f + \beta g) + b\frac{d}{dx}(\alpha f + \beta g) + c(\alpha f + \beta g)$$

by 2

$$= \underline{a\alpha\frac{d^2}{dx^2}f} + a\beta\frac{d^2}{dx^2}g + \underline{b\alpha\frac{d}{dx}f} + b\beta\frac{d}{dx}g + \underline{c\alpha f} + c\beta g$$

(group all the terms w/ α & w/ β)

by 1

$$= \alpha\left(a\frac{d^2}{dx^2}f + b\frac{d}{dx}f + cf\right) + \beta\left(a\frac{d^2}{dx^2}g + b\frac{d}{dx}g + cg\right)$$

by properties of real #'s

$$= \alpha Lf + \beta Lg$$

$\Rightarrow L$ is linear.

6.1 b) $L = \frac{d}{dx} + \ln|\cdot|$ which means $Ly = y' + \ln|y|$.

B/c of the term $\ln|\cdot|$, we suspect the operator is nonlinear. So it is enough to show $L(af) \neq aLf$ $(a \neq 1)$

$L(af) = af' + \ln|af|$

$aLf = a(f' + \ln|f|)$ and b/c

$\ln|af| \neq a\ln|f|$ if $a \neq 1$

L is nonlinear.

6.1 c) $L = \frac{d^2}{dx^2} - \ln x$

$L(af+bg) = \frac{d^2}{dx^2}(af+bg) - \ln x(af+bg)$

by 2

$= a\frac{d^2}{dx^2}f + b\frac{d^2}{dx^2}g - a\ln x f - b\ln x g$

by 1

$= a\left(\frac{d^2}{dx^2}f - \ln x f\right) + b\left(\frac{d^2}{dx^2}g - \ln x g\right)$

(grouping all the terms w/ a & w/ b)

$= aLf + bLg$

$\Rightarrow$ L is linear.

6.2 Thm: Multiplication by a fixed function defines a linear operator.

Proof: Let the function be given. Call it $h(x)$.

Define $Lf = hf$

or equivalently $(Lf)(x) = h(x)\, f(x)$

Then

$$L(af+bg) = h(af+bg)$$

$$= haf + hbg$$

properties of real #'s

$$= ahf + bhg$$

$$= aLf + bLg$$

as needed.

Remark: We used this Thm in Problem 6.1.

6.3

a) $Ly = 0 \quad L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad a,b,c \in \mathbb{R}$

The ODE is linear b/c L is linear.
The ODE is homogeneous, b/c we have
$Ly = g$ where $g \equiv 0$.

b) $Ly = 0 \quad L = \frac{d}{dx} + \ln|\cdot|$

Nonlinear, b/c L is nonlinear.
Homogeneous, b/c $Ly = 0$ $(g \equiv 0)$.

c) $Ly = e^x \quad L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad a,b,c \in \mathbb{R}$

Linear, b/c L is linear.
Inhomogeneous, b/c $Ly = e^x$, so
$g = e^x \neq 0$.

d) $Ly = y \quad L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad a,b,c \in \mathbb{R}$

$$\Leftrightarrow \underbrace{\left(a\frac{d^2}{dx^2} + b\frac{d}{dx} + (c-1)\right)}_{= \tilde{L}} y = 0$$

We could show $\tilde{L}$ is linear
(same proof as in 6.1 w/
$c \to (c-1)$)

$\Rightarrow$ The ODE is linear
& homogeneous.
$(\tilde{L}y = 0,\ \tilde{L}$ is linear & $g = 0)$

6.3e) $Ly = \ln|y| + \ln x \qquad L = \frac{d}{dx} + \ln|\cdot|$

$\iff \frac{d}{dx} y + \ln|y| = \ln|y| + \ln x$

$\iff \frac{d}{dx} y = \ln x$

$\Rightarrow$ linear $\tilde{L} = \frac{d}{dx}$

inhomogeneous $g = \ln x \not\equiv 0$

6.4

a) $y'' - y = 0 \qquad L = \frac{d^2}{dx^2} - 1 \qquad Ly = 0$

linear & homogeneous

b) $y' = \ln x \qquad L = \frac{d}{dx} \qquad Ly = \ln x$

linear & inhomogeneous : $g = \ln x$

c) $y' + (\sin x) y - 1 = 0 \qquad L = \frac{d}{dx} + \sin x$

$g = 1$

$Ly = 1$

linear & inhomogeneous

d) $y'' + by' + cy = y^2$

nonlinear & homogeneous

$(y'' + by' + cy - y^2 = 0)$

e) $y'' + x^2 y' - xy = e^x$ linear & inhomogeneous

$L = \frac{d^2}{dx^2} + x^2 \frac{d}{dx} - x \qquad g = e^x$