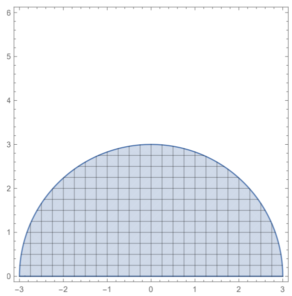
A lamina's mass-density is a measure of its mass per unit area. Consider the following lamina with a density that varies across the object:

The density at any point on a semi-circular lamina D of radius 3 is proportional to the distance from the origin.

1. Draw a partition that creates a mesh of rectangles with equal area ΔA across the lamina D.

   Solution: 

2. Write a function $\rho(x, y)$ to represent the mass-density of the lamina at any point (x, y).

   Solution: $\rho(x, y) = k\sqrt{x^2 + y^2}$ where k is some constant.

3. What does $\rho(x, y)\Delta A$ represent?

   Solution: $\rho(x, y)$ units are mass per unit area. ΔA is an area for one rectangular partition of the lamina D. The product $\rho(x, y)\Delta A$ gives us the mass of one rectangular partition of the lamina D.

4. Explain why the double integral $\iint_D \rho(x, y)\, dA$ represents the mass of the lamina.

   Solution: $\rho(x, y)\Delta A$ gave us the mass of one rectangular partition of the lamina D. The double integral allows us to calculate the sum of an infinite number of areas for small rectangular partitions being multiplied by the density function $\rho(x, y)$. This gives us the total mass of the lamina D.

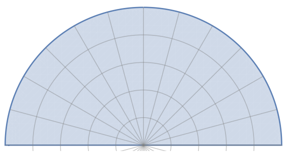
5. Write a double integral that can be evaluated to calculate the mass of the lamina D.

   Solution: $\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} k\sqrt{x^2 + y^2}\, dy\, dx$

6. Evaluate your double integral using Mathematica.

   Solution: In[1]=Integrate[k*Sqrt[x^2 + y^2], {x, -3, 3}, {y, 0, Sqrt[9 - x^2]}]
   Out[1]=$9k\pi$

7. Since our lamina is semi-circular in shape polar coordinates is a more fitting coordinate system. Draw a partition that creates a mesh using polar coordinates with area ΔA across the lamina D.

Solution: 

8. How might we calculate the area for each of the partitions?

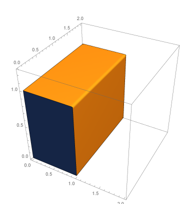
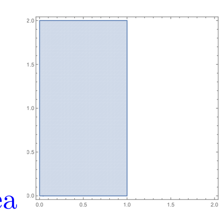
Solution: Using the Jacobian we can transform ΔA from the rectangular coordinate area $\Delta x \Delta y$ to polar coordinate area $r \Delta r \Delta \theta$.

9. Now use polar coordinates to calculate the mass of the lamina D.

Solution: $\int_0^{\pi} \int_0^{3} kr^2 \, dr \, d\theta = 9k\pi$

If our lamina had a density $\rho(x, y) = 1$, then $\iint_D 1 \, dA$ gives the __mass__ of the lamina, and the __area__ of the lamina.

10. We see double integrals can be used to represent multiple ideas. The double integral $\int_0^2 \int_0^1 1 \, dx \, dy$ can represent both a volume and an area. Draw both the solid whose volume and the region whose area is represented by this double integral.

Solution: This double integral could represent the solid  or the area 

We want to be able to apply the concept of multiple integrals to represent a wide variety of ideas across a range of applications (beyond the small number that we cover in this course). To do this we need to be able to think how multiple integrals relate to a situation. We will now explore a few different contexts, and explain why multiple integrals would be useful.

11. **Moment of inertia** is the name given to rotational inertia (i.e. the resistance of a body to changes in rotational motion). The moment of inertia of a particle of mass m about an axis is defined to be $I = mr^2$, where r is the distance from the particle to the axis.

    (a) Given a lamina occupies the region $D = [0,2] \times [0,2]$ with density $\rho(x,y) = 1 + 0.1x$. Explain why we can apply the concept of double integrals to compute the moment of inertia about the x-axis and the moment of inertia about the y-axis.

    Solution: Since we define moments of inertia $I = mr^2$, to find the moment of inertia of the lamina about the x-axis we just need a way to calculate the mass of the lamina at all points (x,y) and multiply by the distance from these (x,y) points to the x-axis.

    Recall $\rho(x,y)\Delta A$ gave us the mass of one rectangular partition of the lamina D. Then to compute the moment of inertia about the x-axis we just need to multiple by y^2 since y is the distance from the x-axis. So $I = y^2\rho(x,y)\Delta A$ is the moment of inertia at one point of the lamina. We need to do this for all points on the lamina, and now integrals come into play. So we have the moment of inertia about the x-axis to be

    $$I_x = \iint_D y^2\rho(x,y)\,dA$$

    Similarly, the moment of inertia about the y-axis is

    $$I_y = \iint_D x^2\rho(x,y)\,dA$$

    (b) Determine if it is more difficult to rotate the lamina about the x-axis or the y-axis.

    Solution: $I_x = \iint_D y^2\rho(x,y)\,dA = 5.86667$ and $I_y = \iint_D x^2\rho(x,y)\,dA = 6.13333$. Therefore, there is a greater resistance to rotational motion if we tried to rotate about the y-axis.

We've extended our concept of integration from single integrals to double integrals. There is no reason to think we cannot keep extending to triple integrals, and how we can apply these integrals in various contexts.

12. Recall from Problem 10 we could think $\iint_D 1\,dA$ could represent with the volume of a solid over a region R or the area of a region R. With this in mind, what could the triple integral $\iiint_S 1\,dV$ represent? Can we draw $\iiint_S 1\,dV$?

    Solution: The triple integral could either represent the four dimensional space the hyper-solid occupies or the volume of the solid S. Unfortunately, we cannot draw four dimensional objects. So we cannot draw the hyper-solid represented by the triple integral. However, we could draw the solid S in three space.

13. The **charge-density** of a solid measures the electric charge (Coulombs) per unit volume of space. If $\sigma(x, y, z)$ represents the charge-density of a solid S, explain what is represented by $\iiint_S \sigma(x, y, z)\,dV$.

    Solution: The charge-density of a solid measures the electric charge (Coulombs) per unit volume of space. The triple integral allows us to calculate the sum of an infinite number of volumes for small rectangular prism partitions being multiplied by the charge-density function $\sigma(x, y, z)$. This gives us the total charge of the solid S.