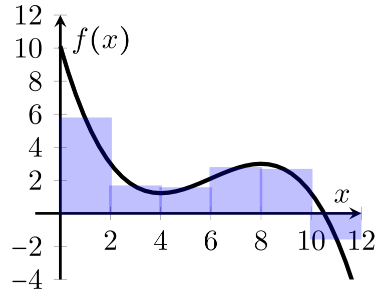
Review of Calculus 1:

Recall from single variable calculus that we are able to approximate the area under continuous curves on closed intervals by partitioning the intervals into n subintervals of equal width Δx to form Riemann sums.



To find the exact area, we refine our partition by adding more and more subintervals letting the width of each subinterval approach zero, resulting in the definite integral.

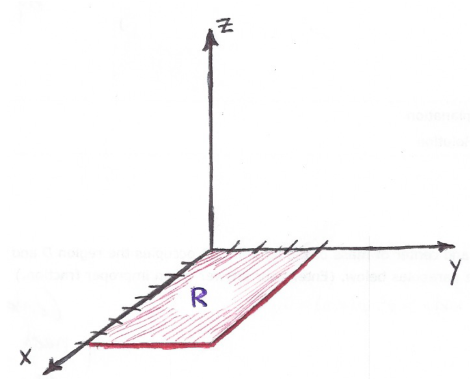
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

Extending Concept to Calculus 3:

We now extend this idea to integrals of functions of two variables, we approximate the volume under a continuous surface on a closed planar region.

1. Draw the planar region defined by $R = [0,8]\times[0,4] = \left\{(x,y)\in\mathbb{R}^2 | 0\le x\le 8, 0\le y\le 4\right\}$ on $z = 0$ in 3-space.

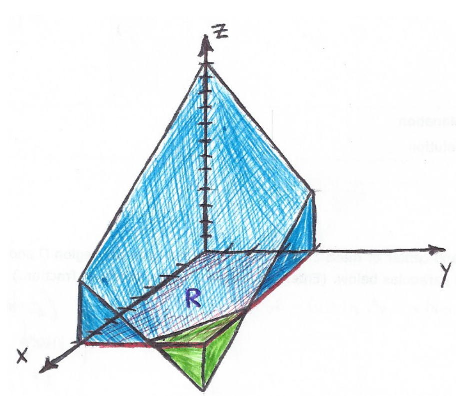
Solution:



Now we approximate the signed volume of the region under a continuous function of two variables and above the closed, bounded region R.

2. Draw the solid represented by $\displaystyle\iint_R 12 - x - 2y\,dA$.

Solution:



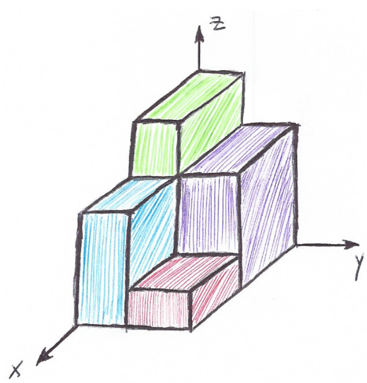
3. Partition the region R into four congruent rectangles with the vertices $(0,0),(0,2),(0,4),(4,0),(8,0),(4,2),(8,2),(4,$ and $(8,4)$.. Use this partition to approximate the signed volume represented by $\iint_R 12 - x - 2y\, dA$ with Riemann sums.

   Solution: We can partition the region R into four rectangles $R_1 = [0,4] \times [0,2]$, $R_2 = [0,4] \times [2,4]$, $R_3 = [4,8] \times [0,2]$, and $R_4 = [4,8] \times [2,4]$. To find the heights for these rectangular prisms (or rectangular cylinders) we will find the z at each lower left corner. So we have $z_1 = 12 - 0 - 2(0) = 12$, $z_2 = 12 - 0 - 2(2) = 8$, $z_3 = 12 - 4 - 2(0) = 8$, and $z_4 = 12 - 4 - 2(2) = 4$. Then we have the approximate volume multiplying the area of the rectangular bases by the height of the rectangle.

   $$V \approx 8(12 + 8 + 8 + 4) = 256$$

   The volume approximation will different if we use different corners or centers of the rectangles to find the heights for the rectangular prisms.

4. Draw a "rectangular" approximation to illustrate your work from the previous problem.

   Solution:

5. Is the double integral $\iint_R f(x,y)dA$ positive? Explain your reasoning.

   Solution: Yes, the double integral is positive. If we look back to the solid we drew in Problem 2, we can see that there is a portion of the solid that is above the xy-plane than below the xy-plane. Therefore, this double integral should have a positive value. We can check this by computing the double integral

   $$\iint_R 12 - x - 2y dA = 128.$$

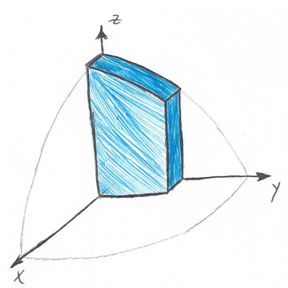
We will explore a graphical interpretation of iterated integrals using the surface $g(x,y) = 16 - x^2 - y^2$ and the region $R = [0,3] \times [0,2]$.

6. By holding x fixed to one value, we can calculate a slice of area $A(x)$ bounded by $g(x,y)$ and R along this x value. Let $x = 0$, and calculate the area bounded by $g(0,y)$ and $[0,2]$.

$$A(0) = \int_0^2 g(0,y)\,dy$$

Solution: $A(0) = \frac{88}{3}$

7. Draw the area that you calculated in the previous problem, and then give it a thickness of Δx to make it a slab of volume $A(0)\Delta x$.

Solution: 

8. Approximate the volume bounded by $g(x,y)$ and R using four volume slabs with a base area $A(x)$ and equal thickness $\Delta x = \frac{3}{4}$.

$$\sum_{i=1}^{4} A(x_i)\Delta x =$$

Solution: $V \approx \frac{3}{4}\left(A(0) + A(\frac{3}{4}) + A(\frac{3}{2}) + A(\frac{9}{4})\right) = \frac{3}{4}\left(\frac{88}{3} + \frac{677}{24} + \frac{149}{6} + \frac{461}{24}\right) = \frac{3821}{32} = 119.40625$

9. To get the exact volume, we need the sum of an infinite number of volume slabs with a base area $A(x)$ and an infinitesimal thickness dx for x from $[0,3]$. Write an integral to represent this volume.

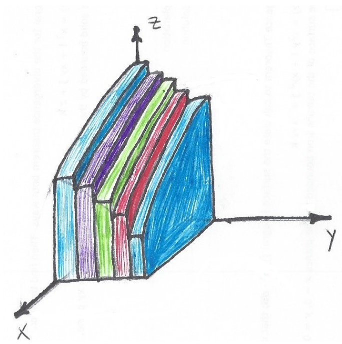
Solution: $\int_0^3 \int_0^2 16 - x^2 - y^2\,dy\,dx$

10. Explain why the double integral

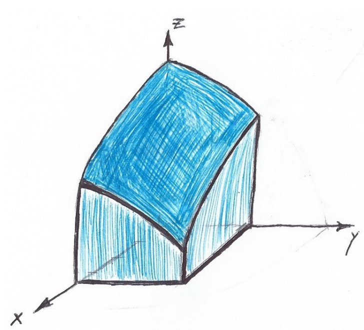
$$\int_0^2 \int_0^3 16 - x^2 - y^2 \, dx \, dy$$

will also determine the exact volume bounded by the surface $g(x, y)$ and region R, and draw a picture to represent it.

Solution: This page first started us computing volume with the iterated integral adding slabs of solids parallel to the y-axis. Now our the order of our iterated integral has reversed, resulting in the addition of slices of solids running parallel to the x-axis. This idea is illustrated in the following picture:

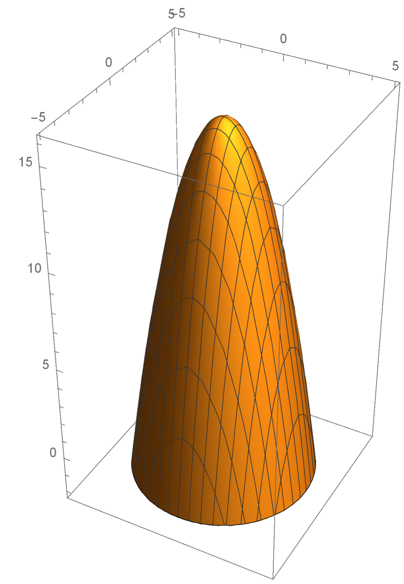


Now returning to the value of the double integral. Notice that the solid represented by the double integral is the following:



Then $\int_0^2 \int_0^3 16 - x^2 - y^2 \, dx \, dy = 70$

11. Shade the solid whose volume is given by the integral $\int_0^4 \int_0^4 16 - x^2 - y^2 \, dx \, dy$.



12. Now write a double integral to represent the volume of the solid bounded by the function $f(x,y) = 16 - x^2 - y^2$ and the xy-plane located in the first octant.

Solution: $\int_0^4 \int_0^{\sqrt{16-x^2}} 16 - x^2 - y^2 \, dx \, dy$

13. Evaluate your integral enough to identify what techniques of integration you need to use to solve.

Solution: This integral will require a combination of trig substitution and u/du substitution. The integration is possible, but it is a bit intensive. There is a better way to calculate this volume using double integrals.

14. Is there another coordinate system that may result in simpler integration?

Solution: By using polar coordinates the integration and bounds are much simpler.

$$\int_0^4 \int_0^{\pi/2} (16 - r^2) r \, d\theta \, dr$$

We need to be careful to include the factor r.