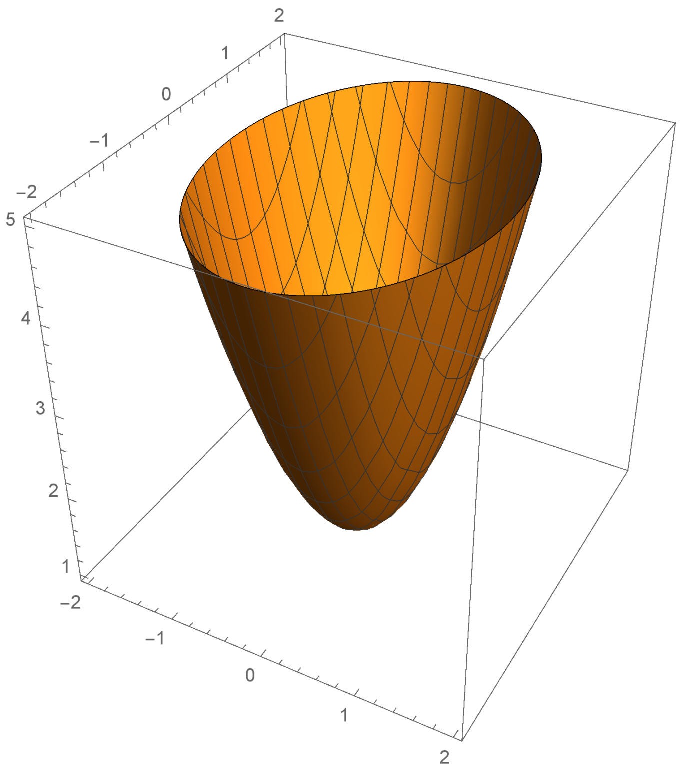
This project reinforces our concept of composition of functions, parameterization, graphing, and differentiation, while gaining insight into why the idea of Lagrange multipliers works.

Our goal is to optimize the function $f(x, y) = 2x^2 + y^2 + 1$ subject to the constraint $g = 0$ where $g(x, y) = x^2 + y^2 - 1$. We will solve this optimization problem using two different methods.

To begin, we graph the surface $f(x, y)$ below:



1. Draw the xyz-axes on the graph.
2. Draw the constraint $g(x, y) = x^2 + y^2 - 1 = 0$ on the xy-plane of the graph.

Notice we are finding extreme values of $f(x, y)$ over a curve, not a region of $f(x, y)$.

Method 1

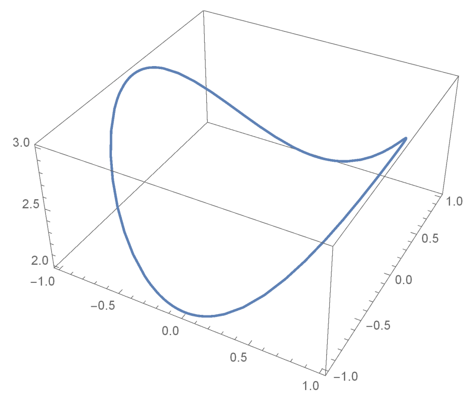
We can optimize $f(x, y) = 2x^2+y^2+1$ subject to the constraint $g = 0$ by first parameterizing the constraint equation.

3. Parameterize the constraint equation $g = 0$ with a planar curve $\mathbf{r}(t)$ in $\mathbb{R}^2$. Be sure to specify the domain for $\mathbf{r}(t)$.

   Solution: $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$, where $0 \le t \le 2\pi$

4. Use Mathematica to graph of $f \circ \mathbf{r}$ to represent the function $f(x, y)$ subject to the constraint $g = 0$. Draw your output below. *Hint: Look up the ParametricPlot3D command on-line.*

   Solution: $f \circ \mathbf{r} = 2\cos^2(t) + \sin^2(t) + 1$. This results in the space curve that looks somewhat like a potato chip:



   Notice that we are finding extrema values of $f(x, y)$ over a curve, not a region of $f(x, y)$.

5. Now using the fact that $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^2$ are differentiable everywhere, write $h = f \circ \mathbf{r}$ just as a function of t, and then use the derivative of h to find the extreme values for the composition function.

   Solution: If $f \circ \mathbf{r} = 2\cos^2(t) + \sin^2(t) + 1$, then $(f \circ \mathbf{r})' = (\nabla f \circ \mathbf{r}) \cdot \mathbf{r}'$ by the Chain Rule. So we have:

$$(f \circ \mathbf{r})' = -4\cos(t)\sin(t) + 2\sin(t)\cos(t).$$

   We set this derivative equal to zero to find the critical points:

$$-4\cos(t)\sin(t) + 2\sin(t)\cos(t) = 0$$

   This gives us that $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. We have four critical points to check for extrema.

   If $t = 0$, then $(x, y) = (1, 0)$. $f(1, 0) = 3$.
   If $t = \frac{\pi}{2}$, then $(x, y) = (0, 1)$. $f(0, 1) = 2$.
   If $t = \pi$, then $(x, y) = (-1, 0)$. $f(-1, 0) = 3$.
   If $t = \frac{3\pi}{2}$, then $(x, y) = (0, -1)$. $f(0, 1) = 2$.

   So we have absolute maxima at $(1, 0, 3)$ and $(-1, 0, 3)$, and absolute minima at $(0, 1, 2)$ and $(0, -1, 2)$.

We can examine the underlying details of Method 1 to find another method to optimize f where $g = 0$.

Method 2

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^2$ be arbitrary functions that are differentiable everywhere.

To find critical points of $f \circ \mathbf{r}$ we set $(f \circ \mathbf{r})' =$ ___0___. Given $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^2$ are differentiable functions everywhere, we can use the Chain Rule to find

$$(f \circ \mathbf{r})' = \underline{\quad \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \quad}.$$

Combining these two ideas we have

$$\underline{\quad 0 \quad} = \underline{\quad \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \quad},$$

which shows the dot product of the vectors ___$\nabla f(\mathbf{r}(t))$___ and ___$\mathbf{r}'(t)$___ are equal to zero. This means that these two vectors are ___orthogonal___.

Recall $\mathbf{r}(t)$ is a parameterization of $g = 0$, so $g(\mathbf{r}(t)) =$ ___0___. If we differentiate both sides we get

$$\underline{\quad \nabla g(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \quad} = \underline{\quad 0 \quad},$$

which shows the dot product of the vectors ___$\nabla g(\mathbf{r}(t))$___ and ___$\mathbf{r}'(t)$___ is equal to zero. This means that these vectors are ___orthogonal___.

We now have $\mathbf{r}'(t)$ is orthogonal to both vectors ___∇f___ and ___∇g___. Since both of these vectors are orthogonal to $\mathbf{r}'(t)$ in $\mathbb{R}^2$, they must be scalar multiples. Therefore, if λ is a constant, we can solve the system ___∇f___ $= \lambda$___∇g___ to find all critical points.

This is called the method of *Lagrange Multipliers.*

Lagrange Multipliers

Suppose $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $g : \mathbb{R}^n \mapsto \mathbb{R}$ are differentiable everywhere. To find the maximum and minimum values of f subject to the constraint $g = k$, where k is some constant [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g = k$]:

(a) Find all values of $\mathbf{x} = (x_1, x_2, \ldots x_n)$ and λ such that

$$\nabla f = \lambda \nabla g$$

and

$$g = k$$

(b) Evaluate f at all the points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ that result from step (a). The largest of these values is the maximum of f; the smallest is the minimum value of f subject to the constraint $g = k$.

6. Use the method of Lagrange multipliers to find all the extreme values of $f(x, y) = 2x^2 + y^2 + 1$ subject to the constraint $g = 0$ where $g(x, y) = x^2 + y^2 - 1$.

Solution:

If $f(x, y) = 2x^2 + y^2 + 1$, then $\nabla f = \langle 4x, 2y \rangle$.
If $g(x, y) = x^2 + y^2 - 1$, then $\nabla g = \langle 2x, 2y \rangle$.
Then using the idea of Lagrange Multipliers we have the equations $\langle 4x, 2y \rangle = \lambda \nabla g$ and $x^2 + y^2 - 1 = 0$.
From these equations we can set up the system of equations:

$$\begin{aligned} 4x &= 2x\lambda \\ 2y &= 2y\lambda \\ x^2 + y^2 - 1 &= 0 \end{aligned}$$

From the first equation we get that either $\lambda = 2$ or $x = 0$ as possible solutions.
From the second equation we get that either $\lambda = 1$ or $y = 0$ as possible solutions.
Note if $\lambda = 2$, then y must equal zero, or if $\lambda = 1$, then x must equal zero.
We can combine this information with the fact that $x^2 + y^2 - 1 =$ to make a conclusion about what points we should check for maximum and minimum values for the function f.
If $x = 0$, then $y = \pm 1$. If $y = 0$, then $x = \pm 1$. So we have four points to check: $(1, 0), (-1, 0), (0, 1), (-1, 0)$.

$$\begin{aligned} f(1, 0) &= 3 \\ f(-1, 0) &= 3 \\ f(0, 1) &= 2 \\ f(0, -1) &= 2 \end{aligned}$$

Using Lagrange Multipliers, we've found the maximum exists at $(1, 0, 3)$ and $(-1, 0, 3)$, and the minimum occurs at $(0, 1, 2)$ and $(0, -1, 2)$.