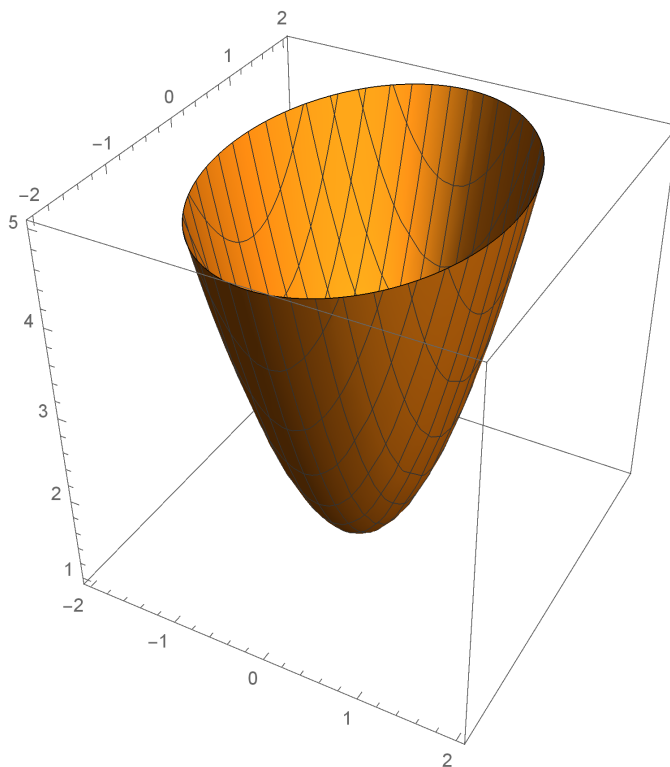


This project reinforces our concept of composition of functions, parameterization, graphing, and differentiation, while gaining insight into why the idea of Lagrange multipliers works.

Our goal is to optimize the function $f(x, y) = 2x^2 + y^2 + 1$ subject to the constraint $g = 0$ where $g(x, y) = x^2 + y^2 - 1$. We will solve this optimization problem using two different methods.

To begin, we graph the surface $f(x, y)$ below:



1. Draw the xyz -axes on the graph.
2. Draw the constraint $g(x, y) = x^2 + y^2 - 1 = 0$ on the xy -plane of the graph.

Notice we are finding extreme values of $f(x, y)$ over a curve, not a region of $f(x, y)$.

Method 1

We can optimize $f(x, y) = 2x^2 + y^2 + 1$ subject to the constraint $g = 0$ by first parameterizing the constraint equation.

3. Parameterize the constraint equation $g = 0$ with a planar curve $\mathbf{r}(t)$ in \mathbb{R}^2 . Be sure to specify the domain for $\mathbf{r}(t)$.
4. Use Mathematica to graph of $f \circ \mathbf{r}$ to represent the function $f(x, y)$ subject to the constraint $g = 0$. Draw your output below. *Hint: Look up the ParametricPlot3D command on-line.*

Notice that we are finding extrema values of $f(x, y)$ over a curve, not a region of $f(x, y)$.

5. Now using the fact that $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^2$ are differentiable everywhere, write $h = f \circ \mathbf{r}$ just as a function of t , and then use the derivative of h to find the extreme values for the composition function.

We can examine the underlying details of Method 1 to find another method to optimize f where $g = 0$.

Method 2

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^2$ be arbitrary functions that are differentiable everywhere.

To find critical points of $f \circ \mathbf{r}$ we set $(f \circ \mathbf{r})' = \underline{\hspace{2cm}}$. Given $f : \mathbb{R}^2 \mapsto \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^2$ are differentiable functions everywhere, we can use the Chain Rule to find

$$(f \circ \mathbf{r})' = \underline{\hspace{2cm}}.$$

Combining these two ideas we have

$$\underline{\hspace{2cm}} = \underline{\hspace{2cm}},$$

which shows the dot product of the vectors $\underline{\hspace{2cm}}$ and $\underline{\hspace{2cm}}$ are equal to zero. This means that these two vectors are $\underline{\hspace{2cm}}$.

Recall $\mathbf{r}(t)$ is a parameterization of $g = 0$, so $g(\mathbf{r}(t)) = \underline{\hspace{2cm}}$. If we differentiate both sides we get

$$\underline{\hspace{2cm}} = \underline{\hspace{2cm}},$$

which shows the dot product of the vectors $\underline{\hspace{2cm}}$ and $\underline{\hspace{2cm}}$ is equal to zero. This means that these vectors are $\underline{\hspace{2cm}}$.

We now have $\mathbf{r}'(t)$ is orthogonal to both vectors $\underline{\hspace{2cm}}$ and $\underline{\hspace{2cm}}$. Since both of these vectors are orthogonal to $\mathbf{r}'(t)$ in \mathbb{R}^2 , they must be scalar multiples. Therefore, if λ is a constant, we can solve the system $\underline{\hspace{2cm}} = \lambda \underline{\hspace{2cm}}$ to find all critical points.

This is called the method of *Lagrange Multipliers*.

Lagrange Multipliers

Suppose $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $g : \mathbb{R}^n \mapsto \mathbb{R}$ are differentiable everywhere. To find the maximum and minimum values of f subject to the constraint $g = k$, where k is some constant [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g = k$]:

- (a) Find all values of $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and λ such that

$$\nabla f = \lambda \nabla g$$

and

$$g = k$$

- (b) Evaluate f at all the points $\mathbf{x} = (x_1, x_2, \dots, x_n)$ that result from step (a). The largest of these values is the maximum of f ; the smallest is the minimum value of f subject to the constraint $g = k$.

6. Use the method of Lagrange multipliers to find all the extreme values of $f(x, y) = 2x^2 + y^2 + 1$ subject to the constraint $g = 0$ where $g(x, y) = x^2 + y^2 - 1$.