Prior projects: Prior to doing this project, students should have done these project:

- Introduction to the types of functions seen in Calc 3
- Composition Practice
- What is the derivative of this thing?

Philosophy behind this project:
In this project, the chain rule is presented from a significantly different perspective than in the textbook. The text treats multiple cases separately, and asks students to memorize formulas rather than the underlying concept. For example, students are not expected to see $\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$ as a dot product. Our approach is more general and conceptual. This allows us to expand the scope of the chain rule that we cover, so students can find the derivative of any valid composition. This more general understanding is required background in both differential equations and differential geometry.

Students are first asked to restate the chain rule from Calc 1. This is the basis for each version of the chain rule they see in this project. We ask them to generalize by seeing the chain rule as "derivative of outside function (evaluated at inside function) multiplied by derivative of inside function." Their job then becomes to figure out what each of the following mean in the context of each problem:

- "derivative of the outside function" (it might be an ordinary derivative, a component-wise derivative of a parameterized curve, a gradient, or a matrix of partials)
- "times" (it might be ordinary multiplication, scalar multiplication, dot product, vector times matrix, matrix times vector, or general matrix multiplication)
- "derivative of the inside function" (it might be an ordinary derivative, a component-wise derivative or a parameterized curve, a gradient, a matrix of partials).

Identifying what "derivative" means in each case is practiced in the "What is the derivative of this thing" project. To substitute the "inside" function into the "derivative of the outside function", they need to have a solid handle on composition, which they studied in the "Composition Practice" project.

Learning Goals:

1. Review of the chain rule for Calc 1
2. Generalize the chain rule to apply in any case of function compositions
3. Apply ideas of differentiation to composition of functions with domains and codomains of varying dimension
4. Incorporate concept of matrices and matrix multiplication to Calc 3 concepts
5. Build students' ability to make conjectures based on their prior understanding
6. Review methods for diagramming composition mappings

Implementation Notes:

1. There should be an explicit connection to the textbook's versions of Chain Rule so that students see where the addition of terms comes from.
2. Matrix of partial derivatives as both a 1×2 matrix and 2×1 matrix appears. As a challenge - some students may want to discuss the transpose.
3. There will need to be a discussion about entry location of matrices.
4. A homework question will ask about the second derivative for one of the problems 6-8.
5. A homework question will ask about the derivative of a composition $\mathbb{R}^2 \mapsto \mathbb{R} \mapsto \mathbb{R}$
6. Page 1 includes a diagram for the mapping of variables for function composition. It may be worth taking a short time to ask students to explain what the diagram represents. These diagrams are helpful for students to understand what the resulting derivative should look like. Students will also be asked to produce their own mapping diagrams.
7. Page 2 scaffolds the students to help them conjecture generalizing the Chain Rule. It may be worth stopping the group work to have a short whole class discussion about the conjecturing process. Make sure students understand why multiplication is now the dot product or matrix multiplication.
8. Page 3 has the students verify their conjecture from page 2. Students then generalize the chain rule for a different composition with less scaffolding. It may be worth to stop group work again to have a whole class discussion about Problem 4(d), asking students to consider if their conjecture is reasonable.
9. Page 4 removes scaffolding, and has students generalize the Chain Rule again. Now as you move through groups, make sure they are discussing if their conjecture is reasonable, "If $\mathbf{h} : \mathbb{R}^2 \mapsto \mathbb{R}^2$, what should its derivative look like?"
10. Page 5 has three concrete examples for students to apply the Chain Rule. Problem 7 requires to take the vector $\mathbf{f}'$, and treat it as a column matrix to carry out the matrix multiplication.

Wrap-Up:

1. While the students are working through Problems 6-8, write on the board the books version of the general version of the Chain Rule:

   The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

   $$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

   for each $i = 1, 2, \ldots, m$.

   If time permits, see if students can make the connection between the work they've been doing through this project and the textbook's version of the Chain Rule. There is a good chance that this will be difficult for the students to make sense of without some assistance. To help them make this connection, have the students pick a random entry in the matrix for their solutions for any problem 6-8. Then have them follow the textbook's formula for calculating this entry. After going through this, there can be a short discussion of the benefits of generalizing the chain rule so that we see the total derivative instead of just a single entry.

2. There should also be some foreshadowing for the students that understanding the Chain Rule will be useful proving other theorems they will see during the semester, such as Lagrange Multipliers.

3. If you have some familiarity of where the chain rule has uses in more advanced mathematics, such as differential geometry, it can spark students interest to provide them with a brief description of where you find it useful.

4. There can be an extension discussion on conjecturing the product rule for multi-variable scalar and vector functions. Students can discuss when it is appropriate to multiply certain functions, and what they expect the product rule to look like. This can help prepare them for the homework assignment of finding the second derivative for one of their chain rule problems.