Prior projects: Prior to doing this project, students should have done the following projects:

- Parametric matching cards

Background content: Prior to doing this project, students should have a working knowledge of the following:

- Given a parameterization for a circle or ellipse, recognize what the graph must look like.
- Given a parameterization for a circle or ellipse, determine the starting point and direction of motion.
- Given the parameterization for a line, recognize that it is a line, be able to graph it by plotting the starting and ending points.
- Create the parameterization for a curve defined by a function $y = f(x)$ by letting $x = t$ and $y = f(t)$.
- Familiarity with cylindrical and spherical coordinates.

Philosophy behind this project:
We find that many students struggle in Calc 3 because they never get solid on creating parameterizations for curves and surfaces. To fill this gap we have two projects. The first is a parameterization matching project. It is similar to one we do in Calc 2, but uses vector notation. This is the second. The graphs on the first page of this project are the same or very similar to graphs on the first matching project. But this time the students must create the parameterization rather than merely recognizing the graphs and matching them. The second page of this project introduces students to parameterization of surfaces. This introduction comes far before the topic first appears in the textbook.

Learning Goals:

1. Whenever creating a parameterization, include the interval of definition for the parameter, even without being prompted for it.
2. Write down parameterizations for a given graph. After completing this project, students should be able to find parameterizations (including intervals) for:
   - Circles and ellipses, starting at any point on the curve and traversing in either direction.
   - Functions defined by $y = f(x)$.
   - Line segments
   - 3D curves with familiar 2D projections.
3. Realize that parameterizing a surface is analogous to parameterizing a curve, with an extra parameter. A curve is parameterized by a single variable so it is essentially one-dimensional. A surface is parameterized by two variables so it is essentially two-dimensional.

Implementation Notes:

1. Before starting, as a warm-up, you can give them the example $x = 4\sin t$, $y = 5\sin t$. Ask them what the graph will look like, and why? Explicitly refer to t as acting like the angle θ from polar coordinates.

2. For the first circle, we're asking them to figure out (or remember) that one way to reverse the direction of motion on an ellipse is to swap the parameterizations for x and y. Other solutions are possible, of course, and students are encouraged to share alternate strategies.

3. For the ellipse, we want them to figure out that changing signs reflects over the appropriate axis, and each reflection changes the direction and possibly the starting point. Another strategy we hope to see students developing is to graph the x coordinate and y coordinate separately as a function of time, and recognize which trig function to use.

4. Line segment: some will remember how to do this, but many may get stuck. Ideally (probably requiring the instructor's help), rather than memorizing a formula, they leave with the concept of finding the vector connecting the start to the end (the direction vector), and thinking of $\mathbf{r}(t)$ = starting point + $t \cdot$ direction vector. With this method the parameter $t \in [0, 1]$. You may choose to also teach parameterizing line segments by arclength.

5. The fourth example (parameterizing $y = \sqrt[3]{x}$) is important. The idea of just allowing your independent variable to be equal to the parameter is generalized to two parametres on the second page in example (c).

6. Helix: encourage them to draw the top view of the curve, and use this picture to parameterize x and y. Next, ask "What does it mean that the pitch (the gap between the coils) is constant?" Asked differently: It looks like the height z is rising at a constant rate. What function (of t) would model that behavior? Add-on question after they solve that problem: How would the graph change if $z = t^2$? A second add-on question, that hints them towards the last question: How would the graph change if z oscillates up and down rather than just climbing?

7. Hints for the saddle (or outline of a Pringle potato chip): As in the previous example, first draw a top-view and parameterize the x and y coordinates separately. Try demonstrating using a coffee cup (hopefully it's fairly cylindrical) and draw on it to show that the x and y coordinates are moving around in a circle (on the cylinder) whilst the z-coordinate oscillates up and down. How many times does the z coordinate need to oscillate up and down each time we go around the circle?

8. Suggestions for wrap-up for the parameterization of curves:

   - Name two ways that I can get the direction of motion on an ellipse to be clockwise instead of counter-clockwise.
   - How do I change the starting point of the ellipse?
   - How do I change the speed that I am moving around the ellipse?
   - How do I recognize that the graph of a parameterization is a line? If they respond that the parameterizations for x and y must be linear, then a harder question: Is there any other way to get a line?
   - How do I create the parameterization for a line segment?
   - If I replace t by $2t$ in my parameterization of a line segment, what happens to the domain for t, and how does the nature of the motion change?
   - If I have the formula for a function given to me, how do I parameterize it?

- Why is it useful to draw a top view of the parameterization of a curve in space?

9. Cut them off in the parameterization of curves section after less than half your time is gone. The second page is their first contact with parameterization of surfaces, so leave enough time for it. Begin by giving them time to read the bottom of page 2, and ask and answer questions. They should complete at the very least the first two examples on the second page.

10. The middle graphs are a top-view of the surface. This is practice for spatial visualization, and also very helpful when they first start finding parameterizations.

11. Some students respond well to this: "When parameterizing curves, as the parameter t varies over the whole of its domain, each point on the curve gets traced out. The main idea behind parameterizing surfaces is that as the parameters u and v vary over their domains, the entire surface should get filled up."

12. For the half disk, here's a sample of the process you want to hint them through: Notice you now have three variables to create functions for. Start with z on this example. Yes, it is constant, so what should the formula for z be? Now to find x and y, think about if I just had to parameterize the outer edge of the circle, then it would look like the examples on the previous page, $x = 3\cos t$, $y = 3\sin t$. But I don't need just the edge, I need to fill in the whole disk, so really that 3 out front should be varying from 0 to 3. That's why I need a second parameter r. You can think of this as polar coordinates, with t representing the varying angle, and r representing the varying radius.

13. For the first paraboloid, here is a possible line of reasoning to hint them through: it is easier to start with x and y in this example, since they are similar to the previous example. Once I've "filled up" the disk, then I realize that I need to create z. Unlike the previous example, z is not constant. I need to find a formula for z so that it is the right height for each point in the disk. On the surface of the paraboloid what must the height z be? Oh yeah, $z = x^2 + y^2$. But I'm not supposed to have xs and ys in my parameterization, just rs and θs, which are the parametres. What shall I do about that? Oh yeah, substitute the parameterizations I already figured out for x and y, which will give me z just in terms of r and θ.

14. Second paraboloid: Since the top-view is not a circle, polar coordinates are not the easiest. We know what z is in terms of x and y, so this example is similar to the cube root example on the first page: just let x and y themselves be the parameters.

15. Students will probably not get to the last two surfaces in the allotted time, but they are a great review and comparison of cylindrical and spherical coordinates for them to do as homework. Students who get a firm grasp early in the course on parameterizing curves and surfaces in cartesian, cylindrical and spherical coordinates tend to do better in the final chapter (vector calculus).

16. The last example, the surface of a cylinder, is the most difficult since we don't suggest what to use as the parameters. This can be made easier by telling them to use cylindrical coordinates θ and z as the parameters. This problem will be confusing for most of them since we used r and θ in the first two examples. But this time r is constant so it is not useful as a parameter. So the easiest way to parameterize is to use θ and z of cylindrical coordinates as the two parameters. The difficulty of this problem can be controlled easily by giving them more or less direction.

17. This project is their first exposure to parameterization of surfaces. It's useful to let them know how often parameterization of curves and surfaces will show up again, but that they reappear in the middle of sections covering more difficult concepts, so if they get solid on it earlier, it will pay off later. Our hope is that at this point they are pretty good at parameterization of curves, and get the main concept of parameterization of surfaces.

18. TAs may choose to encourage students to bring their laptops and use Mathematica to check their results.