Math 2400: Calculus III **Divergence, Stokes', and Green's Theorems and the FTC of line integrals**

Part I - Summary of higher-dimensional versions of the Fundamental Theorem of Calculus
Fill in the blanks (assuming appropriate hypotheses are met for the integrands).

1. Given a surface integral of a vector field $\mathbf{F}$ over a surface S, if the surface S is closed, the surface integral is equal to a triple integral of $\operatorname{div}\mathbf{F}$ over the region bounded by the surface. (Divergence Theorem)

2. Given a line integral of a vector field $\mathbf{F}$ over a curve C, if $\mathbf{F}$ is conservative, then the value of the line integral is the difference between f evaluated at the start point and end point of the curve, where $\nabla f = \mathbf{F}$. (FTC for line integrals)

3. Given a line integral of a vector field $\mathbf{F}$ along a curve C, if the curve C is closed, the line integral is equal to a surface integral of $\nabla \times \mathbf{F}$ over *any* orientable surface that has the curve C as its boundary. (Stokes' Theorem)

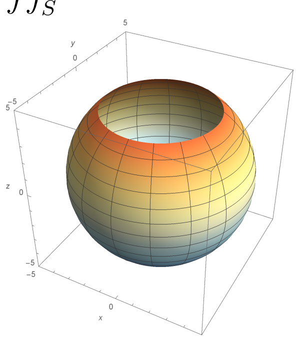
4. Given a line integral of a vector field $\mathbf{F} = \langle P, Q \rangle$ over a planar closed curve C (oriented counter-clockwise), the line integral is equal to a double integral of $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ over the planar region bounded by C. (Green's Theorem)

5. To evaluate $\displaystyle\iiint_E \nabla \cdot \mathbf{F}\, dV$, you can calculate $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the boundary of the solid E. (Divergence Theorem)

6. To evaluate $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ (over an orientable surface S), you can calculate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of the surface S. (Stokes' Theorem)

Part II - Practice problems

1. The figure below shows a surface S, which is a sphere of radius 5 centered at the origin, with the top cut off, so the upper edge of the surface lies at $z = 4$. Use one of the theorems from Chapter 13 to evaluate $\displaystyle\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle 2y, x, z \rangle$.

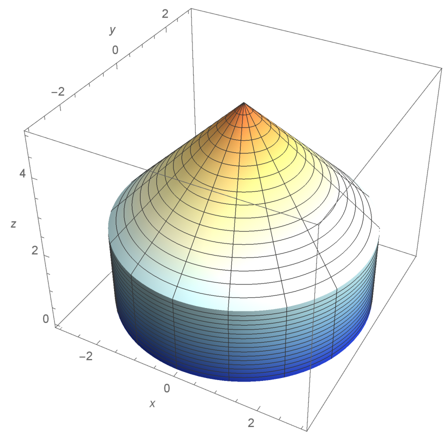


2. Consider the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the closed yurt-shaped surface shown below, and $\mathbf{F} = \langle 3x, 2y, z \rangle$. Notice that the surface comprises three separate pieces: the circular base, the cylinder walls, and the conical top. The cylinder has radius 3 and height 2, and the cone has radius 3 and height 3.

   (a) Discuss with your group the list of steps required to evaluate this surface integral directly.

   (b) Use one of the theorems from Chapter 13 to set up a different type of integral with the same value as the given surface integral.

   (c) Interpret the new integral geometrically to find its value without evaluating it.



3. Consider the two integrals $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle y, -x, 0 \rangle$, and where C_1 is shown below (solid), and C_2 is shown below (dashed). A top-view of the vector field $\mathbf{F}$ is also shown. Do the two line integrals give the same value, or not? Explain.