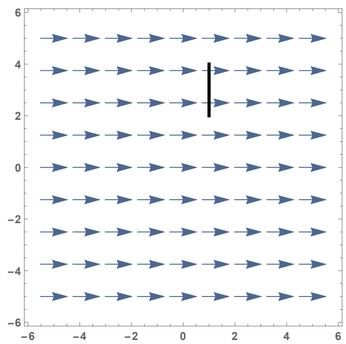
When studying surface integrals over vector fields we often use the word flux. To develop our understanding of flux, we will consider the more intuitive definition of flux as the *total rate of flow across a boundary.*

Suppose we cover the plane with water that is flowing to the right at the rate of 4 ft/sec. We can represent this flowing pool as a vector field **F**. We can use our knowledge of vectors to compute the flux across the line segment from $(1, 2)$ to $(1, 4)$. First, we visualize the flowing pool as a vector field in 2-space.



1. Find and draw **unit** normal vector **n** to the line segment. Choose the direction pointing along the flow. *Notice the unit normal is unitless.*

   Solution: $\mathbf{n} = \langle 1, 0 \rangle$

2. Write the flow rate of the water as a vector **F**. Include appropriate units.

   Solution: $\mathbf{F} = \left\langle 4\frac{\text{ft}}{\text{sec}}, 0\frac{\text{ft}}{\text{sec}} \right\rangle$

3. To compute the amount of flow in the direction of the unit normal vector at each point, find the projection of the flow rate vector **F** onto the unit normal vector **n**. Include appropriate units.

   Solution: Recall the dot product gives us the projection of one vector onto another.

$$\mathbf{F} \cdot \mathbf{n} = \left\langle 4\frac{\text{ft}}{\text{sec}}, 0\frac{\text{ft}}{\text{sec}} \right\rangle \cdot \langle 1, 0 \rangle = 4\frac{\text{ft}}{\text{sec}}$$

   To calculate the flux across a line segment we can use the following idea:

$$\text{Flux}=\text{Flow Rate}\times\text{Length of Segment}$$

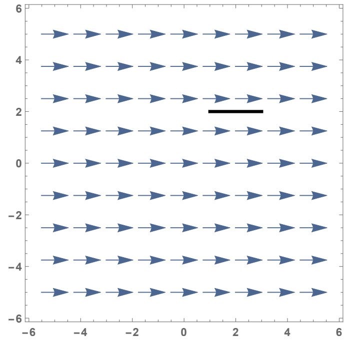
4. Now we multiply the result of this dot product by the length of the line segment to calculate the total rate of water flow across each point of the line segment. Include appropriate units.

   Solution: Let S denote the length of the line segment. Then $S = 2\text{ft}$.

$$(\mathbf{F} \cdot \mathbf{n})S = 4\frac{\text{ft}}{\text{sec}} \times 2\text{ft} = 8\frac{\text{ft}^2}{\text{sec}}$$

   So the total rate flow or *flux* across the line segment from $(1, 2)$ to $(1, 4)$ is $8\frac{\text{ft}^2}{\text{sec}}$

5. Consider the flux across the line segment from $(1, 2)$ to $(3, 2)$.



(a) Use vectors and following the process in the previous page to compute the flux across the line segment. Include units.

$\mathbf{F} =$

Solution: $\langle 4\frac{\text{ft}}{\text{sec}}, 0\frac{\text{ft}}{\text{sec}} \rangle$

$\mathbf{n} =$

Solution: $\langle 0, 1 \rangle$

$\mathbf{F} \cdot \mathbf{n} =$

Solution: $0\frac{\text{ft}}{\text{sec}}$

Segment Length $=$

Solution: 2 ft

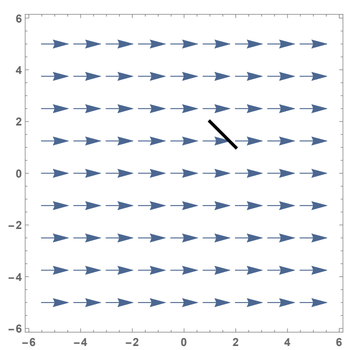
Total Flux $=$

Solution: $(\mathbf{F} \cdot \mathbf{n})(\text{Segment Length}) = 0\frac{\text{ft}}{\text{sec}} \cdot 2\text{ft} = 0\frac{\text{ft}^2}{\text{sec}}$

(b) Explain why your answer to part (a) makes sense intuitively.

Solution: There is no water cross over the line segment.

6. Now consider the flux across the line segment from $(1, 2)$ to $(2, 1)$.



(a) Use vectors to compute the flux across the line segment. Include units.

Solution:

$\mathbf{F} = \langle 4\frac{\text{ft}}{\text{sec}}, 0\frac{\text{ft}}{\text{sec}} \rangle$

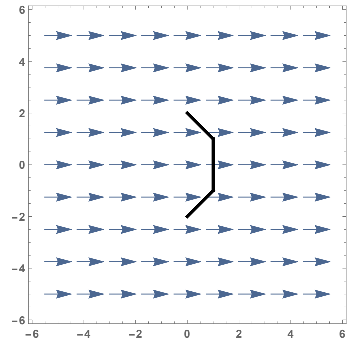
$\mathbf{n} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

Segment Length$= \sqrt{2}$ ft

Total Flux$= (\mathbf{F} \cdot \mathbf{n})(\text{Segment Length}) = \frac{4\sqrt{2}}{2} \cdot \sqrt{2} = 4\frac{\text{ft}^2}{\text{sec}}$

7. Consider the flux across the curve C consisting of line segments from $(0, 2)$ to $(1, 1)$, from $(1, 1)$ to $(1, -1)$, and from $(1, -1)$ to $(0, -2)$.

   (a) Use vectors to compute the flux across the curve C.



Solution: We will find the flux across the curve C by finding the flux for each line segment of the curve, and then add these three values together to get the total flux.

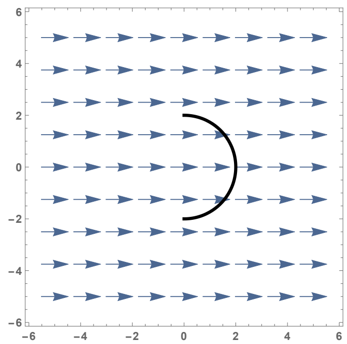
Segment from (0,2) to (1,1):
$\mathbf{F} = \left\langle 4\frac{\text{ft}}{\text{sec}}, 0\frac{\text{ft}}{\text{sec}} \right\rangle$
$\mathbf{n} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
Segment Length= $\sqrt{2}$ ft
Flux Across Segment $s_1 = (\mathbf{F} \cdot \mathbf{n})(\text{Segment Length}) = \frac{4\sqrt{2}}{2} \cdot \sqrt{2} = 4\frac{\text{ft}^2}{\text{sec}}$

Segment from (1,1) to (1,-1):
$\mathbf{F} = \left\langle 4\frac{\text{ft}}{\text{sec}}, 0\frac{\text{ft}}{\text{sec}} \right\rangle$
$\mathbf{n} = \left\langle 1, 0 \right\rangle$
Segment Length= 2 ft
Flux Across Segment $s_2 = (\mathbf{F} \cdot \mathbf{n})(\text{Segment Length}) = 4(2) = 8\frac{\text{ft}^2}{\text{sec}}$

Segment from (1,-1) to (0,-2):
$\mathbf{F} = \left\langle 4\frac{\text{ft}}{\text{sec}}, 0\frac{\text{ft}}{\text{sec}} \right\rangle$
$\mathbf{n} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
Segment Length= 2 ft
Flux Across Segment $s_3 = (\mathbf{F} \cdot \mathbf{n})(\text{Segment Length}) = \frac{4\sqrt{2}}{2} \cdot \sqrt{2} = 4\frac{\text{ft}^2}{\text{sec}}$

Curve C
Total Flux= $4 + 8 + 4 = 16\frac{\text{ft}^2}{\text{sec}}$

8. How might we calculate the flux across the curve C, defined by $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$ with domain of $t \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ in our flowing pool of water?



To help think about this problem, consider the following problems.

(a) Find the unit normal vector to the curve for all points along the curve. *Hint: The unit normal vector is perpendicular to the unit tangent vector.*

Solution: The curve C has a tangent vector $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t)\rangle$ at each time t. Then the we have a unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \langle -\sin(t), \cos(t)\rangle$. Then a vector perpendicular to the unit tangent vector is $\mathbf{T}'(t) = \langle -\cos(t), -\sin(t)\rangle$. Note in this case this vector is a unit vector. Therefore, we have a unit normal vector $\langle -\cos(t), -\sin(t)\rangle$. Also observe that for each time t this normal vector points inward (in the opposite direction of the water flow). To find the normal vector which has an outward direction we can negate it to get $\mathbf{n} = \langle \cos(t), \sin(t)\rangle$

(b) To compute the amount of flow in the direction of the unit normal vector at one point, take the dot product of your unit normal vector and the flow vector.

Solution: $\mathbf{F} \cdot \mathbf{n} = \langle 4, 0\rangle \cdot \langle \cos(t), \sin(t)\rangle = 4\cos(t)$

(c) How can we multiply the result of this dot product by an infinitesimal length of the curve to find the rate of flow across a piece of the curve?

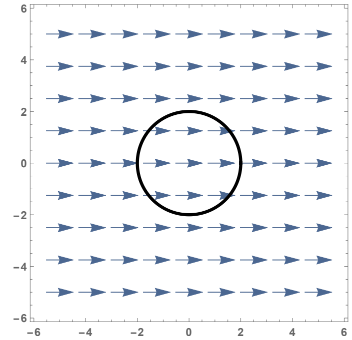
Solution: Recall to calculate the total length of a curve we can use the idea of arc length:
$S = \int_C ds = \int_a^b \sqrt{(x'(t))^2 + ((y'(t))^2}\, dt$
To calculate the flux at each small segment of the curve C, multiply $\mathbf{F} \cdot \mathbf{n}$ by each infinitesimal length, ds, of the curve C: $(\mathbf{F} \cdot \mathbf{n})ds$

Then to calculate the total flux we apply the concept of integrals: $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$

(d) Find the total flow rate across the curve.

Solution:
$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{-\pi/2}^{\pi/2} 4\cos(t)\sqrt{(-2\sin(t))^2 + (2\cos(t))^2}\, dt = \int_{-\pi/2}^{\pi/2} 8\cos(t)\, dt = 16\frac{\text{ft}^2}{\text{sec}}$
Notice that we used two different paths to get from (0,2) to (0,-2) in Problem 7 and Problem 8, and have the same flux. Observe that the vector field is conservative, so we have path independence.

9. Calculate the flux across the closed curve $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$ with domain of $t \in [0, 2\pi]$ in our flowing pool of water? *Note: We can also use a vector form of Green's Theorem to calculate this.*



Solution: Since the vector field is conservative and we have a closed path, we can conclude that the flux across the curve C is zero.

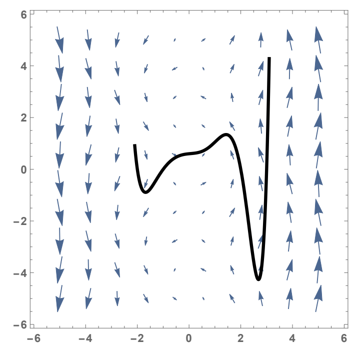
We can also calculate this using Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, dx = \iint_D \nabla \cdot \mathbf{F}\, dA$$

D is the closed circular region bounded by the curve C. Since this is a circular region, we will use polar coordinates. In this case $0 \le r \le 2$, $0 \le \theta \le 2\pi$, and $dA = r\, dr\, d\theta$. Since $\mathbf{F} = \langle 4, 0\rangle$, $\nabla \cdot \mathbf{F} = 0$

$$\int_0^2 \int_0^{2\pi} 0\, r\, d\theta\, dr = 0$$

Now we will generalize this idea for any continuous, smooth curve along a vector field, such as:



In general, given any continuous, smooth curve C along a vector field $\mathbf{F}$ we can calculate the flux, or total rate of flow, across the curve by evaluating the integral

$$\int_C \underline{\quad \mathbf{F} \cdot \mathbf{n}\, ds \quad}$$

10. We now extend this idea to higher dimensions. If we now move from our vector field $\mathbf{F}$ in 2-space to the vector field $\mathbf{G}$ in 3-space, the analogous to our curve C in 2-space is a ____surface____ in 3-space. How might we compute the flux across this boundary in 3-space?

Solution: In 2-space we had the infinitesimal curve length ds. Moving into 3-space, we have the infinitesimal surface area dS. Then we can extend the idea of our work from this project to consider the flux of F across a surface S as:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$