This semester we have extended our concept of integration to include multiple integrals and line integrals over scalar fields and line integrals over vector fields. It is essential to understand the differences between these different types of integration.

1. The following infinitesimals usually appear in integrals when we:

$$ds \qquad dV \qquad dt \qquad d\mathbf{r} \qquad dA \qquad dx \qquad \partial x \qquad Df \qquad dr \qquad dS$$

   (a) $ds,\ dt,\ dx,\ dr$ integrate over a one-dimensional domain.

   (b) ds integrate with respect to arclength.

   (c) $d\mathbf{r}$ take a line integral over a vector field along some curve.

   (d) $dA,\ dS$ integrate over a two-dimensional region.

   (e) dV integrate over a three-dimensional domain.

   (f) $\partial x,\ Df$ does not make sense to use with integration.

As we continue our study of vector calculus it will be important be able to understand and differentiate between line integrals over scalar fields and vector fields.

2. Match the integral to a context that it could represent.

   (i) A $\int_C (2 + x^2 y)\, ds$

   (ii) D $\int_C \mathbf{F} \times d\mathbf{r}$

   (iii) B $\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$

   (iv) B $\int_C \mathbf{F} \cdot d\mathbf{r}$

   (v) D $\int_C \mathbf{F}\, d\mathbf{r}$

   (vi) A $\int_C f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt$

   (vii) B $\int_C \nabla f \cdot d\mathbf{r}$

   (viii) D $\int_C \mathbf{F}\, dx + \int_C \mathbf{F}\, dy + \int_C \mathbf{F}\, dz$

   (ix) B $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$

   (x) B $\int_C P\, dx + \int_C Q\, dy + \int_C R\, dz$

   (xi) D $\int_C 1 d\mathbf{r}$

   (A) A line integral over a scalar field which we can use to find the area of a fence with a curved base C or the mass of a curved wire with varying density.

   (B) A line integral over a vector field which we can use to find the work done moving along the path C.

   (C) Gives the length of a curve.

   (D) Does not make sense.

3. Line Integral over Scalar Field

Before starting the next problem, go to the Wikipedia page on line integrals. Watch the animation on line integrals over scalar fields.

The two questions below each include three graphs that are different representations of the same integral:

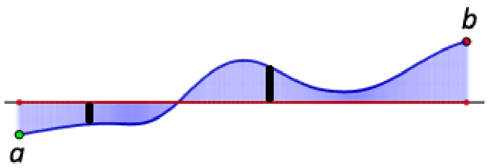
(I) A 2D area representing the total value accumulated along a curve over a scalar field. $\left(\int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt\right)$

(II) A graph of a 2D scalar field and curve C. $\left(\int_C f\,ds\right)$

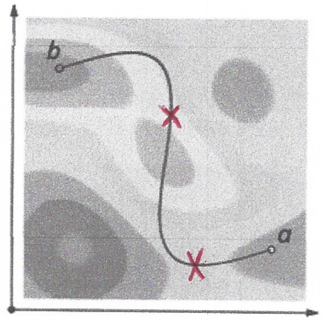
(III) A 3D representation of a scalar field and path along the scalar field above a curve C. $\left(\int_C f\,ds\right)$

(a) Mark an "X" on graph (II) and shade rectangles on graph (III) that correspond to the black rectangles in graph (I).

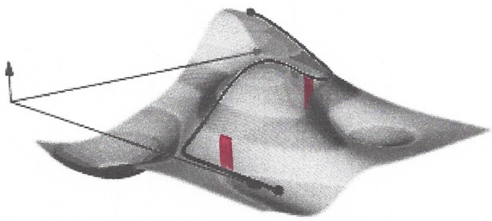
(I) $\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt$:



(II) $\int_C f\,ds$:

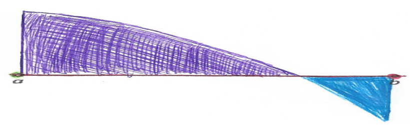


(III)

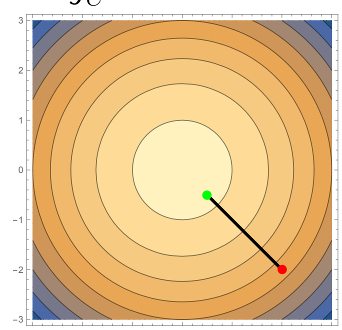


(b) Sketch an area on the axis in graph (I) to represent the values the curve C over the scalar field depicted in graphs (II) and (III).

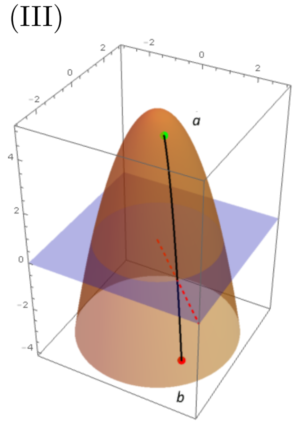
(I) $\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt$:



(II) $\int_C f\,ds$:



(III)



4. Line Integral over Vector Field

Before starting the next problem, go to the Wikipedia page on line integrals. Watch the animation on line integrals over vector fields.

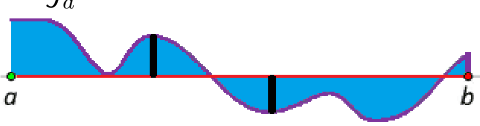
The two questions below each include two graphs that are different representations of the same integral:

(I) A 2D area representing the total value accumulated along a curve over a vector field. $\left(\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt\right)$

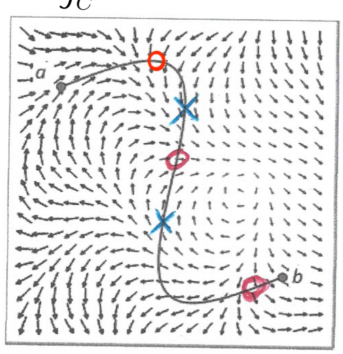
(II) A graph of a 2D vector field and curve C. $\left(\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}\right)$:

(a) Mark an "X" on graph (II) the locations that correspond to the black rectangles in graph (I). Also mark with an "0" on graph (II) the locations that correspond to where graph (I) intersects the axis.

(I) $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$:



(II) $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$:

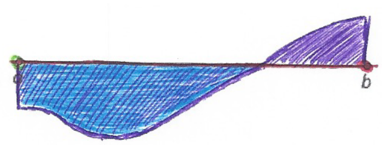


(III) Write your reasoning for the placement of your "X's" and "O's".

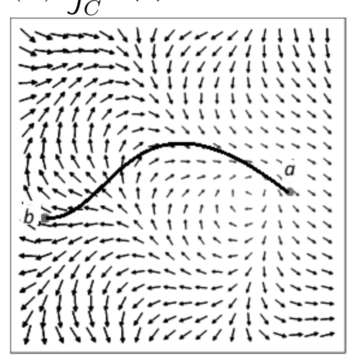
Solution: The first "X" along the path marks where the path moves in the same direction as the vector field, while the second "X" marks where the path is moving in the opposite direction of the vector field. The two "0's" mark where the path is orthogonal to the vector field.

(b) Sketch an area on the axis in graph (I) to represent the dot product of the tangent vectors to the curve C and the field vectors as a particle moves along the curve C from a to b.

(I) $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$:



(II) $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$:



(III) Write your reasoning for your area representation in graph (I).

Solution: The path starts moving in the opposite direction with growing resistance from the vector field as the magnitude of vectors increase. Then the path moves is less opposition to the vector field before becoming orthogonal with the vector field. Finally, the path moves more or less in the same direction as the vector field.