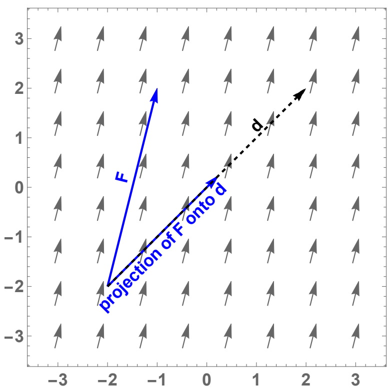
In a previous project we saw examples of using **line integrals over a scalar field** to find the area of a curved fence of varying height, and to find the mass of a curved wire of varying density. We will now learn about **line integrals over a vector field**. A classic application is to find the work done by a force field in moving an object along a curve.

1. **A basic example of calculating work**

   In first-semester calculus we saw the physics formula $W = Fd$, work equals force times distance. This formula fails for movement in a plane or space, because our direction of motion might be at an angle to the force. So in this example we will figure out how to calculate work when the force and the distance are vectors, not scalars.



   In the above graph, the force field is given by the constant function $\mathbf{F}(x, y) = \langle 1, 4\rangle$ and the motion is along the path from $(-2, -2)$ to $(2, 2)$. Call the vector of motion $\mathbf{d}$. Notice that, as usual, the vectors depicting $\mathbf{F}$ are drawn scaled-down to make the graph readable.

   (a) On the graph above, draw one of the vectors of $\mathbf{F}$ to scale, and then draw the vector projection of $\mathbf{F}$ onto the vector $\mathbf{d}$ ($\text{proj}_{\mathbf{d}}\mathbf{F}$).

   (b) Calculate the length of the vector projection of $\mathbf{F}$ onto the vector $\mathbf{d}$. (Recall this is called the **component** of $\mathbf{F}$ in the direction $\mathbf{d}$, written $\text{comp}_{\mathbf{d}}\mathbf{F}$.)

   Solution: The path is $\mathbf{d} = \langle 4, 4\rangle$. The length of the projection of $\mathbf{F}$ into $\mathbf{d}$ is the dot product of $\mathbf{F}$ with the unit vector in the direction of $\mathbf{d}$, or $\mathbf{F} \cdot \dfrac{\mathbf{d}}{|\mathbf{d}|}$. This gives $\langle 1, 4\rangle \cdot \dfrac{\langle 4, 4\rangle}{\sqrt{32}} = \dfrac{20}{\sqrt{32}}$.

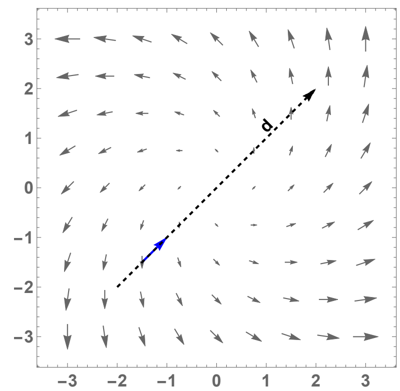
   (c) Now find the work done by the force field $\mathbf{F}$ in moving an object along the vector $\mathbf{d}$.

   Solution: Now that we have the magnitude of the component of the force pointing parallel to $\mathbf{d}$, we can just multiply it by the length of $\mathbf{d}$, giving $\dfrac{20}{\sqrt{32}} \cdot \sqrt{32} = 20$. The sign is positive, indicating that the field did work on the object to move it through the field.

The punch-line of this example:
To find the work done by a force $\mathbf{F}$ in moving an object along a vector $\mathbf{d}$, we first find the component of force $\mathbf{F}$ in the direction $\mathbf{d}$, $\underline{\mathbf{F} \cdot \dfrac{\mathbf{d}}{|\mathbf{d}|}}$. Then multiply this component by the magnitude of $\mathbf{d}$. This gives the formula for work:

$$W = \underline{\mathbf{F} \cdot \mathbf{d}}.$$

2. **Calculating work done when the force field is not constant**
Now we'll find the work done by a force field $\mathbf{F}(x, y) = \langle x - y, x + y\rangle$ in moving an object along the linear path $\mathbf{d}$ from $(-2, -2)$ to $(2, 2)$, The force field and the path are shown below.



(a) Parameterize the path.

$\mathbf{r}(t) = \langle -2 + 4t, -2 + 4t\rangle$

$t \in [\ 0\ ,\ 1\ ]$

(b) Find a vector in the direction of the path that is travelled over a time inteveral of length Δt. Draw this vector on the graph as a typical short segment of the path.

Solution: The path points in the direction $\mathbf{r}'(t) = \langle 4, 4\rangle$. The vector travelled over time Δt is $\mathbf{d} = \langle 4, 4\rangle \Delta t$.

(c) Using the parameterization $r(t)$, substitute to find a formula for $\mathbf{F}$ as a function of t.
Solution: Substituting $x = -2 + 4t$, $y = -2 + 4t$ into the formula for $\mathbf{F}(x, y) = \langle x - y, x + y\rangle$ gives

$$\mathbf{F}(t) = \langle 0, -4 + 8t\rangle$$

(d) Find a formula for the work done by the field in moving an object along the path for a time interval of length Δt (at an arbitrary value of t).
Solution: This is the dot product of the answers to the previous two questions, or

$$\langle 0, -4 + 8t\rangle \cdot \langle 4, 4\rangle\, \Delta t = 4(-4 + 8t)\, \Delta t$$

(e) Add (integrate) the work done along each of the short segments above to find the total work.

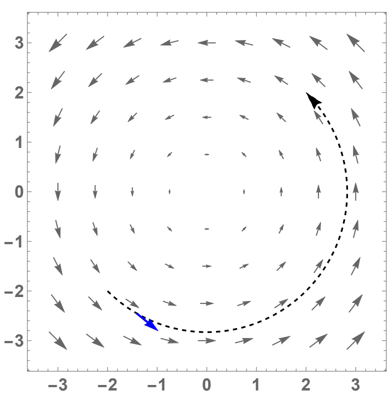
Solution: $\displaystyle\int_0^1 4(-4 + 8t)\, dt = 0.$

(f) Explain intuitively why your answer makes sense.
Solution: Both the field and the path are symmetric about the origin, so the work done heading toward the origin and heading away from the origin cancel each other.

The punch-line of the previous example:
To find the work done by a force field $\mathbf{F}(x, y)$ in moving an object along a vector $\mathbf{d}$, we

- Find $\mathbf{r}(t)$, a parameterization of the ___path of motion of the object___,
- find ___$\mathbf{r}'(t)$___, the tangent vector or direction of the path,
- substitute the ___parameterization___ into the formula for $\mathbf{F}(x, y)$
- find the work along a short segment of the path by taking the ___dot product___ of ___the force___ and ___the tangent vector___, and finally
- ___add (integrate)___ the work along each short section of the path.

3. **Calculating work done when the force field is not constant and the path is not linear**
Now we'll find the work done by a force field $\mathbf{F}(x, y) = \langle -y, x\rangle$ in moving an object along a semi-circular counter-clockwise path from $(-2, -2)$ to $(2, 2)$ (the radius is $2\sqrt{2}$). The force field and the path are shown below.



As before, we'll break the path up into many short segments, find the work done in moving the object along each short segment, then integrate to find total work done.

(a) By looking at the graph and noticing how the path travels through the force field, predict the sign of the work done.

Solution: The curve is going roughly in the direction of the field for the entire path, so we expect the work to be positive.

(b) Parameterize the curve.

$\mathbf{r}(t) = \langle 2\sqrt{2}\cos t, 2\sqrt{2}\sin t\rangle$

$t \in [\ -\frac{3\pi}{4}\ ,\ \frac{\pi}{4}\ ]$

(c) Find a vector in the direction of the path that is travelled over a time inteveral of length Δt. Draw this vector on the graph as a typical short segment of the path. (Note that we are looking for $\Delta \mathbf{r}$, and that $\mathbf{r}'(t) \approx \frac{\Delta \mathbf{r}}{\Delta t}$.)

Solution: $\mathbf{r}'(t) = \langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t\rangle$. We are looking for $\mathbf{d} = \Delta \mathbf{r} = \mathbf{r}'(t)\Delta t = \langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t\rangle \Delta t$.

(d) Using the parameterization $r(t)$, substitute to find a formula for $\mathbf{F}$ as a function of t.
Solution: Substituting $x = 2\sqrt{2}\cos t$, $y = 2\sqrt{2}\sin t$ into the formula for $\mathbf{F}(x, y) = \langle -y, x\rangle$ gives

$$\mathbf{F}(\mathbf{r}(t)) = \langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t\rangle$$

(e) Find a formula for the work done in moving along a segment of the path of length Δt.

Solution: This is the dot product of the answers to the previous two questions, which simplifies to $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\Delta t = \langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t\rangle \cdot \langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t\rangle\,\Delta t = 8(\sin^2 t + \cos^2 t)\,dt = 8\,\Delta t$

(f) Integrate to find the total work done.

Solution: $W = \displaystyle\int_{-3\pi/4}^{\pi/4} 8\,dt = 8\pi.$

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C parameterized by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along** C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t) \cdot \mathbf{r}'(t)\,dt.$$

As shown in Problem 3, the steps in calculating this line integral over a vector field are:

- Find a parameterization $\mathbf{r}(t)$ of the curve C. Give the appropriate interval for t.
- Find $\mathbf{r}'(t)$.
- Substitute the parameterization into the field $\mathbf{F}$.
- Take the dot product of the last two results to get the work done over a short time interval.
- Take the integral of the work done over short time intervals from $t = a$ to $t = b$.