You recently learned how to find the area of a surface by parameterizing, then evaluating the appropriate integral. The first exercise is a review of that concept. In the second problem we will generalize the idea of surface area, introducing a new type of integral: surface integrals of scalar fields.

1. Find the surface area of the part of the surface $z^2 = 4x^2 + 4y^2$ lying between $z = 0$ and $z = 2$.

   (a) Find the intersection of the surface $z^2 = 4x^2 + 4y^2$ and $z = 2$.

   (b) Graph the surface we are trying to find the area of.

   (c) Parameterize the surface using cylindrical coordinates to get $\mathbf{r}(r, \theta)$. Don't forget to include the intervals for the parameters.

   (d) A rectangle with dimensions Δr and $\Delta\theta$ gets mapped by the parameterization onto the surface, to a patch that is roughly a parallelogram. The area of this patch is approximately calculated by the formula $\Delta S =$ ____________________. Calculate the area ΔS for this surface.

   (e) Find the total surface area by integrating.

2. In an integral for surface area, we use the differential $|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$, which geometrically represents:

3. A student asks "We have calculated $\iint_R |\mathbf{r}_u \times \mathbf{r}_v|\ du\, dv$, which is a double integral. I thought a double-integral gives a volume, so why does this integral only give an area?" Respond to this student.

We have seen that the area of a parameterized surface $\mathbf{r}(u, v)$ over the region R can be found by calculating

$$\iint_R 1\,|\mathbf{r}_u \times \mathbf{r}_v|\; du\, dv.$$

The idea of a surface integral is to generalize by replacing the "1" with an arbitrary function.

4. Suppose the surface of problem 1 has a variable density of $\rho(x, y, z) = \sqrt{4 - z^2}$. Find its total mass. Assume the units of mass are grams, and the units of distance are meters.

   (a) What are the units of ρ?

   (b) In the previous problem, we calculated the area of a patch as $|\mathbf{r}_r \times \mathbf{r}_\theta|\,\Delta r\,\Delta\theta =$ ________________.

   (c) What is the mass of a patch?

   (d) Integrate to find the total mass.

The surface integral of a function $f(x, y, z)$ (i.e., a scalar field) over a surface S is written $\iint_S f(x, y, z)\, dS$. It is computed by __________________ the surface S as $\mathbf{r}(u, v)$ and computing __________________________, (where R defines the domain of u and v).

5. Practice at home: Re-do the integrals in problems 1 and 4 using spherical coordinates. Make sure you get the same numerical answers as when you did the calculations in cylindrical coordinates.

6. Practice at home: Use spherical coordinates to find the area of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies above the plane $z = 2$. Then find the mass of that surface if the density is given by $\rho = z$.