Prior projects: Prior to doing this project, students should have completed the FTC fill-in-the-blank activity, and possibly also the FTC practice activity.
Background content: This is one of the capstone activities for the course. In order to successfully complete the activity, students must have mastered the entire content of the course. Before beginning they should have internalized the summary on page 973 of the text.
Philosophy behind this project:
The intention of this project is for students to think deeply and carefully about how and when the various higher-dimensional versions of the Fundamental Theorem of Calculus apply. There are 8 cards that students are asked to break into groups that have equal values. The cards are constructed to be similar so that matching can only be done by carefully applying the theorems.

Learning Goals:

1. Surface integrals can be converted to line integrals using Stokes' Theorem if the integrand of the surface integral is the curl of a vector field.

2. A vector field $\mathbf{G}$ is the curl of another vector field if $\text{div}\mathbf{F} = 0$

3. If $\text{div}\mathbf{F} = 0$, then a surface integral $\iint_S \mathbf{F} \cdot d\mathbf{r}$ is equal to the surface integral over any other surface that shares the same boundary with S.

4. Line integrals can be converted into surface integrals using Stokes' Theorem, provided the curve is closed.

5. Triple integrals can be converted into surface integrals using the Divergence Theorem if the integrand of the triple integral is the divergence of a vector field.

6. Surface integrals can be converted into triple integrals using the Divergence Theorem, provided the surface is closed.

7. If a field is conservative, then the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ can be calculated using the Fundamental Theorem for Line integrals by taking the difference of the potential function at ending and starting points. This means that the value of a line integral in a conservative field depends only on the endpoints (the boundary of the path), not the path itself.

8. A field $\mathbf{F}$ can be determined to be conservative by finding a potential function for it. In other words, if $\mathbf{F} = \nabla f$ for some scalar field f, then $\mathbf{F}$ is conservative.

9. A field $\mathbf{F}$ with sufficiently smooth components is conservative as long as $\text{curl}\mathbf{F} = 0$ (See Theorem 4 in Section 13.5)

10. A consequence of the Fundamental Theorem for Line Integrals, is that when C is a closed curve and $\mathbf{F}$ is a conservative field, then $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$. We can see this by realizing that since the curve is closed, the starting and ending points are the same, so the difference in the potential function at the end and start points is 0.

Implementation Notes: The project is designed to be printed then cut into 8 cards. Students are asked to put the cards into groups that have equal results. The students might be tempted to pair the cards, but for the correct answer, some of the groups have three cards.
Answers and explanations:

1. A tricky point of this project is to figure out which Theorem to try to apply, then which fields to take the divergence of or curl of, and which fields to match those functions with.

2. Cards C, F and G have the same value. To see that each of C and F are the same as G, first notice that the curl of $\langle -2xy, x^2, 1\rangle$ is $\langle 0, 0, 4x\rangle$. Thus directly applyng Stokes' Theorem gives that Card G is the same as Card C and that Card G is the same as Card F. Matching is made more confusing by the fact that the vector fields all have the name $\mathbf{F}$. It is up to the students to realize that in order to apply Stokes', we need for the curl of one vector field to be equal to the other vector field, regardless of their names.

3. There are issues regarding orientation of the normal vector for cards C, F and G. Evaluation of the surface integrals in Cards C and F requires a choice of orientation. We generally choose an outward normal vector for closed surfaces. However, the surfaces shown are not closed. If we think of each of them as part of a closed surface by covering their tops with a disk, then the outward normal vector points downwards. In this interpretation, using the right-hand-rule, the positive orientation of the boundary curve travels clockwise as viewed from above. This would mean that the value on Card G would be the opposite of cards C and F. The surface, however, is not closed. So a conventional choice for orientation of the normal vector is upwards. In this choice of orientation the positive orientation of the boundary curve is counter-clockwise, and the values of Cards C and G (and F and G) are equal.

4. Cards C and F are the same because they are both equal to Card G. However, it is also possible to see directly that Cards C and F are the same, as follows. A consequence of Stokes' Theorem is that if a field is the curl of another field, then a surface integral can be evaluated by looking only at the boundary of the surface. Thus the surface integral of any two surfaces that have the same boundary is identical (this should be reminiscent of path-independence for conservative fields in that only the boundary matters). So determining that Cards C and F are the same requires only to show that the field $\mathbf{F} = \langle 0, 0, 4x\rangle$ is the curl of some vector field. If $\mathbf{F} = \text{curl}\mathbf{G}$ for some field $\mathbf{G}$, then $\text{div}\mathbf{F} = \text{div}(\text{curl}\mathbf{G}) = 0$ by Theorem 11 of Section 13.5. So if $\text{div}\mathbf{F}$ must be 0 in order for $\mathbf{F}$ to be the curl of some field. This turns out to be a sufficient condition as well. In this example, we see that the divergence of $\langle 0, 0, 4x\rangle$ is zero, thus $\langle 0, 0, 4x\rangle$ is the curl of some vector field. Thus Stokes' Theorem applies, the surface integral depends only on the boundary, and the two cards have equal values.

5. Cards D and A are equal. Since $\text{div}\langle 2xy, x^2, 1\rangle = 2y$, a direct application of the Divergence Theorem shows these two cards are equal. The graphs are deceiving, and must in interpreted in the context of the accompanying text. The graph on card D shows a ball, which is a solid, or region in space. The graph on Card A is the boundary of that region, which is a surface.

6. Cards B, E and H have the same value. The field on Card B is conservative since its curl is 0 (apply Theorem 4 in Section 13.5). Thus, since C is a closed curve, the value of the line integral is 0. Alternately, we can see that the field on Card B is conservative by confirming that it is the gradient of the potential function on Card E, again implying that the value of the line integral is 0.