Some motivational problems in number theory

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For each problem, indicate whether you think the answer is YES or NO. Then indicate how hard you think it is: (E)asy, (M)edium, (H)ard, (U)nsolved.

0.1. **Are there infinitely many primes?**

0.2. **Is there a closed formula for the n-th prime?** Let's say a function of n a calculus student could read.

0.3. **Is there a (possibly multivariate) polynomial that gives only primes when evaluated on all integer inputs?**

0.4. **Are there infinitely many primes of the form $4n + 1$?**

0.5. **If you know the nth prime ends in a particular digit, is the final digit for the $(n + 1)$-st prime equally likely to be any of the four possibilities $1, 3, 7, 9$?**

0.6. **Are there infinitely many primes of the form $n^2 + 1$?**

0.7. **Are there infinitely many primes p for which $p + 2$ is prime?**

0.8. **Up to N, are there always more natural numbers with an odd number of prime factors than with an even number of prime factors?**

0.9. **Does $x^2 - 1141y^2 = 1$ have any solutions?** Note: if you ask the computer to check up to 25 digits, it will find none.

0.10. **Does $x^3 - y^2 = 1$ have any integer solutions besides $(1, 0)$?**

0.11. **Does $x^n + y^n = z^n$ have any integer solutions for integers $n > 2$?**

0.12. **Is there an algorithm to determine if a given polynomial equation in any number of variables has an integer solution?**

0.13. **Are there any quadratic forms with integer coefficients which represent all positive integers?**

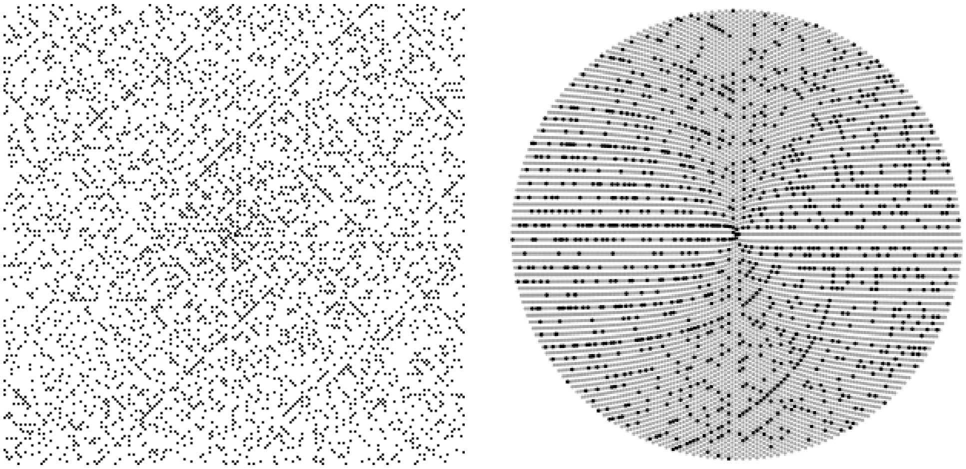
0.14. **Does there exist a deterministic polynomial-time algorithm (in the number of digits) to determine if n is prime?**

0.15. **Can we factor numbers in deterministic polynomial time?**

0.16. **For any irrational number α, are there infinitely many rational numbers p/q such that $|\alpha - p/q| < 1/q^2$?**

0.17. **Given n, if it is even, divide by 2 and if it is odd, return $3n+1$; if we iterate this rule, must we eventually reach the loop $1 \to 4 \to 2 \to 1$?**

0.18. **Given a real number $\alpha > 0$, if it is > 1, then subtract 1 and if it is < 1, then invert it; if we iterate this rule, must we eventually reach a loop?**



Images from Wikipedia.

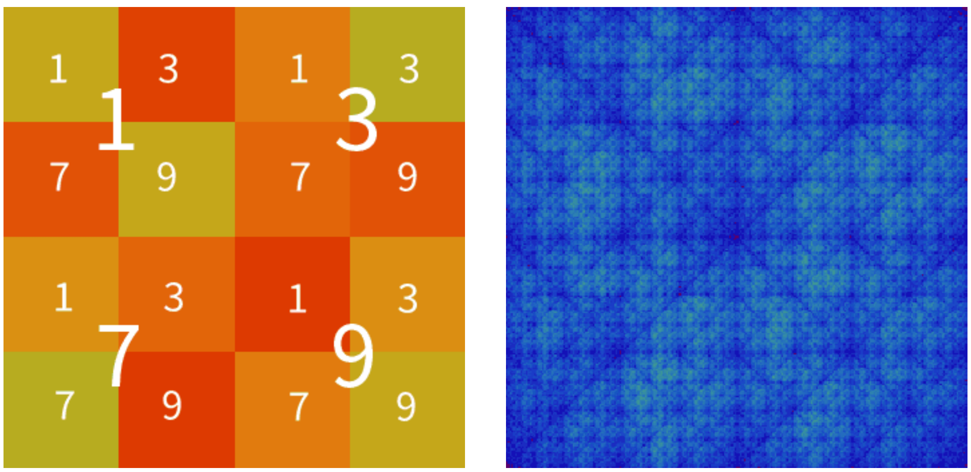


Image from matthen.com.

$$
\begin{aligned}
&(k+2)(1-[wz+h+j-q]^2-[(gk+2g+k+1)(h+j)+h-z]^2-\\
&\qquad [16(k+1)^3(k+2)(n+1)^2+1-f^2]^2-[2n+p+q+z-e]^2-\\
&\qquad [e^3(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2-\\
&\qquad [16r^2y^4(a^2-1)+1-u^2]^2-[n+l+v-y]^2-\\
&\qquad [(a^2-1)l^2+1-m^2]^2-[ai+k+1-l-i]^2-\\
&\qquad [((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2]^2-\\
&\qquad [p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2-\\
&\qquad [q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2-\\
&\qquad\qquad [z+pl(a-p)+t(2ap-p^2-1)-pm]^2)
\end{aligned}
$$