Badges Quiz Synthesis Questions

December 15, 2016

Name:

1 Short answer questions

1. (Synthesis I) How many injective functions are there from A to B, when $|A| = n$ and $|B| = m$?

   Soln. If $m < n$, there are none (by pigeonhole principle, where the inputs are pigeons and the outputs are holes). If $m \geq n$, then the first input can be taken to any of m outputs, while the second input can be taken to any of the remaining $m-1$ outputs, etc. There are n inputs, so we obtain

   $$m \cdot (m-1) \cdot \cdots \cdot (m-n+1) = \frac{m!}{(m-n)!}.$$

2. (Synthesis I) A function $f : A \to A$ can be thought of as a type of relation. Specifically, R is a relation on A where aRb if and only if $f(a) = b$. Suppose, for this problem, that $A = \mathbb{Z}$. If possible, give an example of a function, which, when considered as a relation, is reflexive. If not possible, explain why.

   Soln. Unravelling definitions, $f : \mathbb{Z} \to \mathbb{Z}$ is reflexive as a relation if $f(x) = x$ for all x. Therefore the identify function works, and no other function works.

3. (Synthesis II) As a followup to the previous question, can you give an example of a non-injective, non-surjective function $f : \mathbb{Z} \to \mathbb{Z}$ whose corresponding relation is transitive?

   Soln. Unravelling definitions, $f : \mathbb{Z} \to \mathbb{Z}$ is transitive if

   $$f(x) = y \wedge f(y) = z \implies f(x) = z.$$

   Rephrasing,

   $$f(f(x)) = z \implies f(x) = z.$$

   In other words, every w in the range satisfies $w = f(w)$. In other words, the entire range is fixed by f. A nice example of such a function is the identify function, but it is injective and surjective. Another simple example of such a function is a constant function, e.g. $f(x) = 1$, which is neither surjective nor injective. Try drawing a diagram to help you digest this.

4. (Synthesis I) Consider the set X of all equivalence classes of finite boolean expressions in variables P and Q (i.e. things like $P \wedge Q$ and $\sim P \implies Q$ etc.), under the relation of logical equivalence. What is the cardinality of this set?

   Soln. The logical equivalence class of a boolean expression is determined by its truth table, i.e. its values in the pairs $(P, Q) = (T, T), (T, F), (F, T), (F, F)$. There are two choices for each of the four values, so there are $2^4 = 16$ equivalence classes. Said another way, there are 16 possible truth tables in two variables.

5. (Synthesis II)

   Let X be the set of finite subsets of $\mathbb{N}$ (here, $\mathbb{N}$ is the positive integers). In set builder notation,

   $$X = \{U \subset \mathbb{N} : |U| < \infty\}.$$

   Note that X is infinite. Give an explicit function $f : X \to X$ which is surjective but not injective. (By *give an explicit*, I mean give enough information so that I can compute the function on any input.)

   Soln. There are many functions that work, but here's a simple one I thought of. Let $f : X \to X$ be given by the following rule: for an input set X, remove its largest element and return the set that remains. For example, $f(\{2, 3, 4\}) = \{2, 3\}$. Every set can be formed from a larger set by eliminating some larger number, so f is surjective. But $f(\{2, 3, 4\}) = f(\{2, 3, 5\})$ so f is not injective.

6. (Synthesis I) Consider a function $f : A \to B$. Put a relation on A as follows:

$$xRy \iff f(x) = f(y).$$

This is an equivalence relation. If $|A| = 7$ and $|B| = 3$, how many equivalence classes could the relation have? (There are finitely many possibilities, depending on f. List them.)

Soln. The equivalence classes are

$$[x] = \{a \in A : f(a) = f(x)\}.$$

This means that the elements of A are grouped according to where f takes them. In other words there is one equivalence class for each element of the range of f. Therefore the question is asking how many elements could the range of f have? It must have at least 1 element, but can have as many as $|B| = 3$. Therefore the possibilities are:

$$1, 2, 3.$$

7. (Synthesis II)

Suppose $f : F \to Z$ and $g : G \to Z$ are functions. We define the *fibre product* as follows:

$$F \times_Z G := \{(x, y) \in F \times G : f(x) = g(y)\}.$$

(a) Find functions $f : \mathbb{Z} \to \mathbb{R}$ and $g : \mathbb{Z} \to \mathbb{R}$ so that the resulting fibre product $\mathbb{Z} \times_{\mathbb{R}} \mathbb{Z}$, considered as a relation on $\mathbb{Z}$, is *has the same sign as*.

(b) What choice of Z will result in $F \times_Z G = F \times G$, the Cartesian product?

Soln. The fibre product is a standard notion in research mathematics, but it is a little wonky to get used to at first (like everything).

(a) Suppose

$$f(x) = g(x) = \text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$$

Then, we have

$$F \times_Z G = \{(x, y) \in F \times G : \text{sign}(x) = \text{sign}(y)\}$$

which is what was desired.

(b) Suppose $Z = \{1\}$ or any set with one element. Then the functions f and g are necessarily constant functions. We obtain

$$F \times_Z G = \{(x, y) \in F \times G : f(x) = g(y)\} = \{(x, y) \in F \times G : 1 = 1\} = \{(x, y) \in F \times G\},$$

as desired.