Combinatorial Proof Examples

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A combinatorial proof is a proof that shows some equation is true by explaining why both sides count the same thing. Its structure should generally be:

Explain what we are counting.

Explain why the LHS (left-hand-side) counts that correctly.

Explain why the RHS (right-hand-side) counts that correctly.

Conclude that both sides are equal since they count the same thing.

1. Give a combinatorial proof that

$$\binom{n}{k} = \binom{n}{n-k}.$$

   (a) What are we counting?
   We are counting how many ways a subset of k things can be chosen from n things.

   (b) How does the left side count this?
   The symbol $\binom{n}{k}$ counts this by definition.

   (c) How does the right side count this?
   To choose a subset of k things, it is equivalent to choose $n-k$ things to exclude from the subset. There are $\binom{n}{n-k}$ ways to do this.

Now here is a complete theorem and proof.

Theorem 1. *Suppose $n \geq 1$ is an integer. Suppose k is an integer such that $1 \leq k \leq n$. Then*

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof. We will explain that both sides of the equation count the number of ways to choose a subset of k things from n things (and they must therefore be equal).

The left side counts this by definition.

To choose a subset of k things, it is equivalent to choose $n-k$ things to exclude from the subset. There are $\binom{n}{n-k}$ ways to do this. Therefore the right side also counts this.

Hence both sides are equal. □

2. Give a combinatorial proof that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

   (a) What are we counting?

   The number of ways to choose a subset of size k from a set of size n.

   (b) How does the left side count this?

   By definition.

   (c) How does the right side count this?

   Choose an element from the set of size n, and call it x. We condition on whether x is in the chosen subset.

   Case 1: If x is to be included in the chosen subset, then there are $\binom{n-1}{k-1}$ ways to complete the subset.

   Case 2: If x is to be excluded from the chosen subset, then there are $\binom{n-1}{k}$ ways to complete the subset.

   Adding the counts from the two cases gives the total number of ways to choose the subset.

Here is a complete theorem and proof.

Theorem 2. *Suppose $n \geq 1$ is an integer. Suppose k is an integer such that $1 \leq k \leq n$. Then*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. We will demonstrate that both sides count the number of ways to choose a subset of size k from a set of size n.

The left hand side counts this by definition.

To see that the right hand side also counts this, choose an element from the set of size n, and call it x. We condition on whether x is in the chosen subset.

Case 1: If x is to be included in the chosen subset, then there are $\binom{n-1}{k-1}$ ways to complete the subset.

Case 2: If x is to be excluded from the chosen subset, then there are $\binom{n-1}{k}$ ways to complete the subset.

Adding the counts from the two cases gives the total number of ways to choose the subset. This gives the right hand side of the equation.

Therefore both sides enumerate the same quantity and must therefore be equal. $\square$