The Sandbox puzzle

Katherine Stange, Math 2001, CU Boulder

The Sandbox puzzle

Katherine Stange, Math 2001, CU Boulder

1. Suppose we have an infinitely deep sandbox.

The Sandbox puzzle

Katherine Stange, Math 2001, CU Boulder

1. Suppose we have an infinitely deep sandbox.
2. If the sand is level with the box, that's "zero" sand.

The Sandbox puzzle

Katherine Stange, Math 2001, CU Boulder

1. Suppose we have an infinitely deep sandbox.
2. If the sand is level with the box, that's "zero" sand.
3. If we excavate sand, we have negative sand.

The Sandbox puzzle

Katherine Stange, Math 2001, CU Boulder

1. Suppose we have an infinitely deep sandbox.
2. If the sand is level with the box, that's "zero" sand.
3. If we excavate sand, we have negative sand.
4. If we pile up sand, we have positive sand.

The Sandbox puzzle

Katherine Stange, Math 2001, CU Boulder

1. Suppose we have an infinitely deep sandbox.
2. If the sand is level with the box, that's "zero" sand.
3. If we excavate sand, we have negative sand.
4. If we pile up sand, we have positive sand.
5. You begin with zero sand, and can remove or add sand by the 6 or 15-gallon pailful.

The Sandbox puzzle

Katherine Stange, Math 2001, CU Boulder

1. Suppose we have an infinitely deep sandbox.
2. If the sand is level with the box, that's "zero" sand.
3. If we excavate sand, we have negative sand.
4. If we pile up sand, we have positive sand.
5. You begin with zero sand, and can remove or add sand by the 6 or 15-gallon pailful.
6. What amounts of sand are possible to obtain?

Some thoughts from the class discussion

1. We can get 0 by not doing anything.

Some thoughts from the class discussion

1. We can get 0 by not doing anything.
2. We can get 6 by adding one 6-gallon scoop.

Some thoughts from the class discussion

1. We can get 0 by not doing anything.
2. We can get 6 by adding one 6-gallon scoop.
3. We can get 9 by adding a 15-gallon scoop and taking away a 6-gallon scoop ($12 = 15 - 6$).

Some thoughts from the class discussion

1. We can get 0 by not doing anything.
2. We can get 6 by adding one 6-gallon scoop.
3. We can get 9 by adding a 15-gallon scoop and taking away a 6-gallon scoop ($12 = 15 - 6$).
4. We can get 3 by adding a 15-gallon scoop and taking away two 6-gallon scoops ($3 = 15 - 2 \cdot 6$).

Some thoughts from the class discussion

1. We can get 0 by not doing anything.
2. We can get 6 by adding one 6-gallon scoop.
3. We can get 9 by adding a 15-gallon scoop and taking away a 6-gallon scoop ($12 = 15 - 6$).
4. We can get 3 by adding a 15-gallon scoop and taking away two 6-gallon scoops ($3 = 15 - 2 \cdot 6$).
5. We can get -6 by taking away a 6 gallon scoop.

Some thoughts from the class discussion

1. We can get 0 by not doing anything.
2. We can get 6 by adding one 6-gallon scoop.
3. We can get 9 by adding a 15-gallon scoop and taking away a 6-gallon scoop ($12 = 15 - 6$).
4. We can get 3 by adding a 15-gallon scoop and taking away two 6-gallon scoops ($3 = 15 - 2 \cdot 6$).
5. We can get -6 by taking away a 6 gallon scoop.
6. It looks like we can get anything that 'looks like' $15x + 6y$.

Your task is to prove this theorem

Theorem

In the sandbox puzzle, the amount of sand that are possible to obtain are exactly those which are a multiple of 3 *gallons (positive, zero or negative).*

Important: This actually says two things:

1. We can get every multiple of 3.
2. Anything we get is a multiple of 3.

So the proof may need to prove these two things separately.

Proof.

First, since 6 and 15 are both multiples of three, any amount of sand formed by adding and subtracting them will still be a multiple of three.

Proof.

First, since 6 and 15 are both multiples of three, any amount of sand formed by adding and subtracting them will still be a multiple of three.

We can get 0 gallons of sand by doing nothing.

Proof.

First, since 6 and 15 are both multiples of three, any amount of sand formed by adding and subtracting them will still be a multiple of three.

We can get 0 gallons of sand by doing nothing.

We can form 3 gallons of sand by adding one 15-gallon scoop and then removing two 6-gallon scoops.

Proof.

First, since 6 and 15 are both multiples of three, any amount of sand formed by adding and subtracting them will still be a multiple of three.

We can get 0 gallons of sand by doing nothing.

We can form 3 gallons of sand by adding one 15-gallon scoop and then removing two 6-gallon scoops.

We wish to show we can form any other amount of sand that is a multiple of 3 gallons. But if we wish to have $3n$ gallons of sand, for some positive integer n, then we repeat the process above exactly n times.

Proof.

First, since 6 and 15 are both multiples of three, any amount of sand formed by adding and subtracting them will still be a multiple of three.

We can get 0 gallons of sand by doing nothing.

We can form 3 gallons of sand by adding one 15-gallon scoop and then removing two 6-gallon scoops.

We wish to show we can form any other amount of sand that is a multiple of 3 gallons. But if we wish to have $3n$ gallons of sand, for some positive integer n, then we repeat the process above exactly n times.

We have now demonstrated how to form any positive integer multiple of three gallons.

Proof.

First, since 6 and 15 are both multiples of three, any amount of sand formed by adding and subtracting them will still be a multiple of three.

We can get 0 gallons of sand by doing nothing.

We can form 3 gallons of sand by adding one 15-gallon scoop and then removing two 6-gallon scoops.

We wish to show we can form any other amount of sand that is a multiple of 3 gallons. But if we wish to have $3n$ gallons of sand, for some positive integer n, then we repeat the process above exactly n times.

We have now demonstrated how to form any positive integer multiple of three gallons.

And negatives? We can form -3 gallons of sand by *subtracting* one 15-gallon scoop and then *adding* two 6-gallon scoops.

Proof.

First, since 6 and 15 are both multiples of three, any amount of sand formed by adding and subtracting them will still be a multiple of three.

We can get 0 gallons of sand by doing nothing.

We can form 3 gallons of sand by adding one 15-gallon scoop and then removing two 6-gallon scoops.

We wish to show we can form any other amount of sand that is a multiple of 3 gallons. But if we wish to have $3n$ gallons of sand, for some positive integer n, then we repeat the process above exactly n times.

We have now demonstrated how to form any positive integer multiple of three gallons.

And negatives? We can form -3 gallons of sand by *subtracting* one 15-gallon scoop and then *adding* two 6-gallon scoops.

To obtain $-3n$ gallons of sand, for n a positive integer, we should repeat the above process n times.

Proof.

First, since 6 and 15 are both multiples of three, any amount of sand formed by adding and subtracting them will still be a multiple of three.

We can get 0 gallons of sand by doing nothing.

We can form 3 gallons of sand by adding one 15-gallon scoop and then removing two 6-gallon scoops.

We wish to show we can form any other amount of sand that is a multiple of 3 gallons. But if we wish to have $3n$ gallons of sand, for some positive integer n, then we repeat the process above exactly n times.

We have now demonstrated how to form any positive integer multiple of three gallons.

And negatives? We can form -3 gallons of sand by *subtracting* one 15-gallon scoop and then *adding* two 6-gallon scoops.

To obtain $-3n$ gallons of sand, for n a positive integer, we should repeat the above process n times.

This shows how to obtain any multiple-of-3 number of gallons of sand. □

Proof.

Since 6 and 15 are multiples of 3, any sand amount formed by adding and subtracting them will still be a multiple of 3.

Proof.

Since 6 and 15 are multiples of 3, any sand amount formed by adding and subtracting them will still be a multiple of 3.

It will be useful to have some notation for this problem. If we can write an integer n as a sum $n = 15x + 6y$ for some integers x and y, then we will have created n gallons of sand, by adding x 15-gallon scoops and y 6-gallon scoops.

Proof.

Since 6 and 15 are multiples of 3, any sand amount formed by adding and subtracting them will still be a multiple of 3.

It will be useful to have some notation for this problem. If we can write an integer n as a sum $n = 15x + 6y$ for some integers x and y, then we will have created n gallons of sand, by adding x 15-gallon scoops and y 6-gallon scoops.

Note that if x is negative, then "adding x scoops" really means "substracting $-x$ scoops".

Proof.

Since 6 and 15 are multiples of 3, any sand amount formed by adding and subtracting them will still be a multiple of 3.

It will be useful to have some notation for this problem. If we can write an integer n as a sum $n = 15x + 6y$ for some integers x and y, then we will have created n gallons of sand, by adding x 15-gallon scoops and y 6-gallon scoops.

Note that if x is negative, then "adding x scoops" really means "substracting $-x$ scoops".

Next, suppose we have a multiple of three, say $3n$ where n is an integer.

Proof.
Since 6 and 15 are multiples of 3, any sand amount formed by adding and subtracting them will still be a multiple of 3.
It will be useful to have some notation for this problem. If we can write an integer n as a sum $n = 15x + 6y$ for some integers x and y, then we will have created n gallons of sand, by adding x 15-gallon scoops and y 6-gallon scoops.
Note that if x is negative, then "adding x scoops" really means "substracting $-x$ scoops".
Next, suppose we have a multiple of three, say $3n$ where n is an integer.
We can form 3 in the following way: $3 = 15 \cdot 1 + 6 \cdot (-2)$ (in other words, taking $x = 1$ and $y = -2$ in our notation).

Proof.

Since 6 and 15 are multiples of 3, any sand amount formed by adding and subtracting them will still be a multiple of 3.

It will be useful to have some notation for this problem. If we can write an integer n as a sum $n = 15x + 6y$ for some integers x and y, then we will have created n gallons of sand, by adding x 15-gallon scoops and y 6-gallon scoops.

Note that if x is negative, then "adding x scoops" really means "substracting $-x$ scoops".

Next, suppose we have a multiple of three, say $3n$ where n is an integer.

We can form 3 in the following way: $3 = 15 \cdot 1 + 6 \cdot (-2)$ (in other words, taking $x = 1$ and $y = -2$ in our notation).

Therefore, multiplying this equation by n, we have

$$3n = 15 \cdot n + 6 \cdot (-2n).$$

Proof.

Since 6 and 15 are multiples of 3, any sand amount formed by adding and subtracting them will still be a multiple of 3.

It will be useful to have some notation for this problem. If we can write an integer n as a sum $n = 15x + 6y$ for some integers x and y, then we will have created n gallons of sand, by adding x 15-gallon scoops and y 6-gallon scoops.

Note that if x is negative, then "adding x scoops" really means "substracting $-x$ scoops".

Next, suppose we have a multiple of three, say $3n$ where n is an integer.

We can form 3 in the following way: $3 = 15 \cdot 1 + 6 \cdot (-2)$ (in other words, taking $x = 1$ and $y = -2$ in our notation).

Therefore, multiplying this equation by n, we have

$$3n = 15 \cdot n + 6 \cdot (-2n).$$

This gives a recipe in scoops of the 6 and 15 gallon pail, to form any $3n$ gallons of sand. □

Proof.

Rephrasing the question, we ask which integers n can be expressed as a linear combination

$$n = 15x + 6y$$

for some integers x and y.

Proof.

Rephrasing the question, we ask which integers n can be expressed as a linear combination

$$n = 15x + 6y$$

for some integers x and y.

Since 3 divides both 6 and 15, it divides any linear combination of them.

Proof.

Rephrasing the question, we ask which integers n can be expressed as a linear combination

$$n = 15x + 6y$$

for some integers x and y.

Since 3 divides both 6 and 15, it divides any linear combination of them.

Conversely, suppose n is a multiple of 3, so that we may write it as $n = 3k$ for an integer k.

Proof.

Rephrasing the question, we ask which integers n can be expressed as a linear combination

$$n = 15x + 6y$$

for some integers x and y.

Since 3 divides both 6 and 15, it divides any linear combination of them.

Conversely, suppose n is a multiple of 3, so that we may write it as $n = 3k$ for an integer k.

Then, since

$$3 = 15 \cdot 1 + 6 \cdot (-2),$$

we have

$$3k = 15 \cdot k + 6 \cdot (-2k).$$

□