Relations and Equivalence Relations

November 2, 2015

1 Relations

What is a relation? Informally, we work on some set S and it is some property any pair of elements of S may or may not have. For example, $a < b$, if elements of S can be compared in size, or $a = b$ if there is a notion of equality.

We can visualize a relation on a set as a bunch of arrows. For example, the relation $<$ on the set $S = \{1, 2, 3\}$ looks like this:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[dl] \\ 3 & \end{tikzcd}$$

We can also write it as a set of ordered pairs, for example, the relation above becomes

$$\{(1, 2), (2, 3), (1, 3)\} \subset S \times S.$$

We actually use the latter as the definition of a relation, mathematically speaking:

Definition 1. *A* relation *on a set A is a subset $R \subseteq A \times A$.*

That's it! It just tells us which elements are related (the pairs in the subset R) and which are not (the pairs not in the subset R).

Fill in the table.

explanation	A	R	arrow diagram
$\leq$ on the integers $0, 1, 2$			
$=$ on all integers			
'comes before in alphabetical order' on the vowels			
'divides' on the natural numbers			
'is a subset of' on all subsets of $\{a, b\}$			

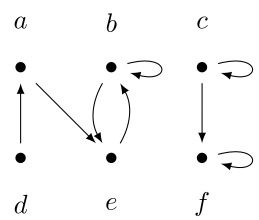
2 Properties of Relations

Definition 2. *Let R be a relation defined on a set A.*

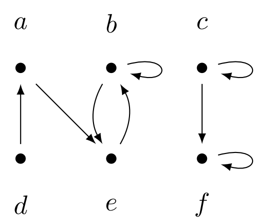
1. *If for all $x \in A$, we have xRx, we call R* reflexive.
2. *If for all $x, y \in A$, we have $xRy \implies yRx$, we call R* symmetric.
3. *If for all $x, y, z \in A$, we have $(xRy \wedge yRz) \implies xRz$, we call R* transitive.

A relation that has all three properties is called an equivalence relation*.*

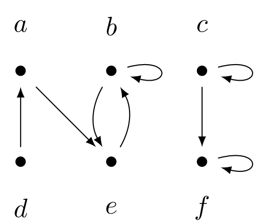
1. What additional arrows would have to be included in the diagram below for the relation to be reflexive? (Draw them in as well.)



2. What additional arrows would have to be included for it to be symmetric? (Draw them.)



3. What additional arrows would have to be included for it to be transitive? (Draw them.)



4. Explain what being *reflexive* means in terms of the arrow diagram.

5. Explain what being *symmetric* means in terms of the arrow diagram.

6. Explain what being *transitive* means in terms of the arrow diagram.

7. On the first page, add a column where you identify whether each one is reflexive, symmetric and/or transitive. Just mark ‘R’, ‘S’ and ‘T’ as appropriate.

8. On the set below, draw a relation that is symmetric, but not transitive. Write out the relation as a set of ordered pairs.

$A\bullet \qquad\qquad \bullet B$

$\bullet C$

9. On the set below, draw a relation that is transitive but not symmetric or reflexive. Write out the relation as a set of ordered pairs.

$A\bullet \qquad\qquad \bullet B$

$\bullet C$

10. On the set below, draw an equivalence relation (reflexive, symmetric and transitive). Write out the relations as a set of ordered pairs.

$A\bullet \qquad\qquad \bullet B$

$\bullet C$

11. Now on the set $\{A, B, C\}$, draw *all possible equivalence relations.*

12. A *partition* of a set S is any way of breaking it up into disjoint subsets whose union is the entire set S. Draw pictures of all possible partitions of $\{A, B, C\}$. What is the relation to the previous problem?