Math 2001: Convergence and Continuity: ϵ and δ (Katherine Stange)

Definition 1. *Let $L_1, L_2, \ldots, L_n, \ldots$ be a sequence of real numbers. We say that the sequence L_n* converges *to a real number L if for every real $\epsilon > 0$, there exists a positive integer N so that $|L_n - L| < \epsilon$ for all $n > N$.*

Definition 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$. We say that f is* continuous at *a if for every real $\epsilon > 0$, there exists a real $\delta > 0$ such that $|f(a) - f(b)| < \epsilon$ whenever $|a - b| < \delta$.*

Note: "If P(x) then Q(x)" can be rephrased as "Q(x) for all x satisfying P(x)" or as "Q(x) whenever P(x)". Do you see an if/then statement in the definitions above?

1. Read the first definition carefully. Taking into account the "Note" above, the definition is of the form

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, P(n) \implies Q(n).$$

   Fill in:

$$P(n) =$$
$$Q(n) =$$

2. Discuss why I write $P(n)$ instead of $P(N)$, $P(\epsilon)$ etc. What is special about the variable n here?

3. Consider the sequence $1, 1/2, 1/3, \ldots$, i.e. $L_n = 1/n$.
   (a) What L does L_n converge to, do you think? In what follows, take L to be that value.

   (b) When $\epsilon = 1$, and $N = 1$, prove that $|L_n - L| < \epsilon$ for all $n > N$.

   (c) When $\epsilon = 1/2$, and $N = 2$, prove that $|L_n - L| < \epsilon$ for all $n > N$.

   (d) When $\epsilon = 1/3$, what N do we need? Prove that $|L_n - L| < \epsilon$ for all $n > N$ for your choice.

   (e) When $\epsilon = 1/100$, what N do we need?

   (f) When $\epsilon = 1/k$, where k is an integer, what N do we need?

   (g) For any ϵ, what N do we need?

(h) Write a beautiful proof that L_n converges to 0.

4. Using the strategies of the previous part, show that $L_n = 1 - 1/\sqrt{n}$ converges to 1.

5. Read the second definition carefully. Taking into account the "Note" above, the definition is of the form

$$\forall \epsilon > 0, \exists \delta > 0, P(b) \implies Q(b).$$

Fill in:

$$P(b) =$$
$$Q(b) =$$

6. Discuss why I write $P(b)$ instead of $P(f)$, $P(\epsilon)$ etc. What is special about the variable b here?

7. Consider the function $f(x) = x^2$.

   (a) Do you think f is continuous at 0?

   (b) When $\epsilon = 1$, and $\delta = 1$, prove that $|f(0) - f(b)| < \epsilon$ whenever $|0 - b| < \delta$.

   (c) When $\epsilon = 1/4$, and $\delta = 1/2$, prove that $|f(0) - f(b)| < \epsilon$ whenever $|0 - b| < \delta$.

   (d) When $\epsilon = 1/9$, what δ do we need? Prove that $|f(0) - f(b)| < \epsilon$ whenever $|0 - b| < \delta$ for your choice.

   (e) When $\epsilon = 1/100$, what δ do we need?

   (f) When $\epsilon = 1/k$, where k is an integer, what δ do we need?

   (g) For any ϵ, what δ do we need?

(h) Write a beautiful proof that f is continous at 0.

8. Prove that $f(x) = x + 2$ is continuous at 3.

Extra problems:

1. Can you write a definition for the notion of $\lim_{x\to\infty} f(x) = L$?
2. Can you write a definition for the notion of $\lim_{x\to a} f(x) = L$?
3. Can you write a definition for the notion of $\lim_{x\to a} f(x) = \infty$?
4. Can you write a definition for the notion of $\lim_{x\to\infty} f(x) = \infty$?
5. Prove some limits! In other words, come up with examples of the above and write the proofs.