MATHEMATICS 2001 – EXISTENCE PROOF PUZZLES!

It's puzzle time! The goal today is to produce some nicely-written proofs.

Theorem 1. *There exists a set of any finite cardinality.*

Correction: There exists a set of any non-negative finite integer cardinality.

Proof. Let n be a non-negative integer. We will create a set that has cardinality n. Let
$$S = \{1, 2, \ldots, n\}.$$
Then $|S| = n$. $\square$

Theorem 2. *Every odd integer is the sum of two consecutive integers.*

Proof. Let n be an odd integer. Then $n = 2m+1$ for some integer m. Then consider the consecutive integers m and $m+1$. Then n is their sum, since
$$m + (m + 1) = 2m + 1 = n.$$
$\square$

Theorem 3. *Every odd integer is the difference between two consecutive perfect squares.*

Proof. Let n be an odd integer. Then $n = 2m + 1$ for some integer m. Consider the consecutive perfect squares m^2 and $(m + 1)^2$. Then their difference is
$$(m + 1)^2 - m^2 = m^2 + 2m + 1 - m^2 = 2m + 1 = n.$$
$\square$

Theorem 4. *There are arbitrarily large gaps in the sequence of prime numbers.*

Proof. To prove this, we will demonstrate a gap of size n. To do so, consider the integer $(n + 1)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot n \cdot (n + 1)$.

Then $(n + 1)!$ is divisible by 2, 3, $\ldots$, n, $n + 1$. Let k be any of these n different integers. Then the integer $m := (n+1)!+k$ is divisible by k, but is bigger than k, hence k is not 1 or m. Hence it is composite. Therefore the n consecutive integers
$$(n + 1)! + 2, (n + 1)! + 3, (n + 1)! + 4, \ldots, (n + 1)! + (n + 1)$$
are all composite. $\square$

Theorem 5. *Between any two irrational numbers, there is a rational number.*

Correction: Between any two distinct irrational numbers, there is a rational number.

Proof. Let α and β be two distinct irrational numbers. Then they each have decimal expansion:
$$\alpha = a + 0.a_0a_1a_2\ldots$$
$$\beta = b + 0.b_0b_1b_2\ldots$$
where $a, b \in$ and the a_i and b_i represent the digits. Note that these expansions must continue forever.

Suppose without loss of generality that $\alpha > \beta$. Then, either $a > b$ or $a = b$. If $a > b$, then a is an integer satisfying $\beta < a < \alpha$, and we are done.

So, suppose that $a = b$. Then, since $\alpha \neq \beta$, at some point their decimal expansions disagree for the first time. That is, $a_i = b_i$ for $i < I$, and then $a_I > b_I$. In that case the rational number $a + 0.a_1a_2\ldots a_I$ lies strictly between α and β. $\square$