Properties of Cardinality

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Definition 1. *Two sets A and B are said to have the* same cardinality *(written $|A| = |B|$) if there exists a bijective function $f : A \to B$. Otherwise, they are said to have different cardinalities (written $|A| \neq |B|$).*

Definition 2. *Let $n \in \mathbb{Z}$. If a set A has the same cardinality as $\{1, 2, 3, \ldots, n\}$, then we say it has* cardinality n *and write $|A| = n$.*

Theorem 1. $|\{a, b\}| = 2$.

Proof. There is a bijection between $\{a, b\}$ and $\{1, 2\}$ given by $f(a) = 1$, $f(b) = 2$. □

Theorem 2. $|\mathbb{Z}| = |\mathbb{N}|$

Note: Even though the first function you may think of, namely $f : \mathbb{N} \to \mathbb{Z}$ given by $f(x) = x$, is *not* a bijection, that doesn't mean there isn't some *other* function that is a bijection.

Proof. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ depicted in this table:

x	$f(x)$
1	0
2	1
3	-1
4	2
5	-2
6	3
7	-3
$\vdots$	$\vdots$

This function has a formula,

$$f(x) = \begin{cases} x/2 & 2 \mid x \\ -(x-1)/2 & 2 \nmid x \end{cases}$$

We claim this function is bijective.

First we show it is injective. Note that it takes odd natural numbers to non-positive integers, and even natural numbers to positive integers. Therefore to rule out collisions, we need only check that the functions $f(x) = x/2$ and

$f(x) = -(x-1)/2$ are injective on even and odd numbers, respectively. I leave each of these as an exercise.

Next, we show it is surjective. But any positive integer n is the image of $2n$ and any non-positive integer $-n$ is the image of $2n+1$. (I leave the details as an exercise.) $\square$

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For this to be a good definition, we expect three properties:

Theorem 3. *For any set A, $|A| = |A|$.*

Theorem 4. *For any sets A and B, if $|A| = |B|$, then $|B| = |A|$.*

Theorem 5. *For any sets A, B and C, if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.*

Prove these.