Proof Quiz #9 Solution

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Honor Code Rules

Proof Quizzes are open book, but are to be completed on your own without collaboration. To be specific, you may use your course notes, textbook, course website resources, course videos. You may not use the internet beyond the course websites. You may not ask anyone else for help (except your professor), including other humans, or posting/entering your question into the internet. You may not share the questions or answers with anyone else.

Have you read, understood, and followed the honor code rules above?

$$YES \quad / \quad NO$$

Please write your best written proof of the following theorem. You will be graded on logic as well as writing.

Theorem 1. *The intervals* $[0,1]$ *and* $[e,\pi]$ *have the same cardinality.*

Note: e here is the natural logarithm.

Proof. To show these two sets have the same cardinality, it suffices to find a bijection between them.

Let us define $f(x) = (\pi-e)x+e$. On the domain $[0,1]$, $0 \leq x \leq 1$. For such x, $e \leq (\pi-e)x+e \leq \pi$. Therefore this defines a function $f : [0,1] \to [e,\pi]$.

Next, we check that f is injective. Suppose $f(x_1) = f(x_2)$. Then $(\pi-e)x_1 + e = (\pi-e)x_2 + e$, which implies $x_1 = x_2$.

Finally, we check that f is surjective. Let $y \in [e,\pi]$. Then let $x = (y-e)/(\pi-e)$. We must check two things. First, we check that $x \in [0,1]$, which is true since $0 \leq y-e \leq \pi-e$. And second, we check that $f(x) = (\pi-e)(y-e)/(\pi-e)+e = y-e+e = y$. □

Proof. By the Cantor-Bernstein-Schröder Theorem, it suffices to find two injections

$$f : [0,1] \to [e,\pi], \quad g : [e,\pi] \to [0,1].$$

We can use the functions

$$f(x) = x/100 + e, \quad g(x) = x - e.$$

Since $100 < \pi - e$, f is well-defined with codomain $[e,\pi]$, i.e. its outputs lie in the interval. Similarly, since $\pi - e < 1$, the second is also well-defined with codomain $[0,1]$. Next, since the functions are linear, they are injective. □

Proof. To show these two sets have the same cardinality, it suffices to find a bijection between them.

Let us define $f(x) = (\pi-e)x+e$. On the domain $[0,1]$, $0 \leq x \leq 1$. For such x, $e \leq (\pi-e)x+e \leq \pi$. Therefore this defines a function $f : [0,1] \to [e,\pi]$.

Next, I claim that this function has an inverse. Define $g(x) = (x-e)/(\pi-e)$. On the domain, $e \leq x \leq \pi$. So $0 \leq x-e \leq \pi-e$. Therefore $0 \leq g(x) \leq 1$. Therefore this defined a function $g : [e,\pi] \to [0,1]$. We will now check this is the inverse of f, by computing

$$g \circ f(x) = ((\pi-e)x+e-e)/(\pi-e) = x, \quad f \circ g(x) = (\pi-e)(x-e)/(\pi-e)+e = x.$$

Since $f : [0,1] \to [e,\pi]$ is invertible, it is bijective, and we are done. □

Some common issues:

1. Forgetting to check that the functions were well-defined (i.e. mapped the domain into the codomain).
2. Forgetting to check, in surjectivity, that the pre-image was in the domain.
3. Insufficient details, e.g. claiming injectivity or inverse with no further detail.
4. Arguing with a graph. To draw conclusions from a graph, you have to prove the picture is accurate. It isn't (it's a sketch), but it probably has important features that are, but each of those has to be proved if you plan to use it anyway, so a graph doesn't save work.
5. The usual error with surjectivity, which is to solve for x before we know it exists, without explanation.

About the final error, suppose we wish to prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 1$ is surjective. Consider these two solutions:

Example Solution 1

Let $y \in \mathbb{R}$. Then $y = x + 1$, so $x = y - 1$.

Example Solution 2

Let $y \in \mathbb{R}$. Let $x = y - 1$. Then $x \in \mathbb{R}$ and $f(x) = (y - 1) + 1 = y$.

Analysis

The problem with *Example Solution 1* is that it assumes x exists, right when it says $y = x + 1$. We don't know that y is of that form yet: that's what we're trying to prove! You can work your way around this with language like "we seek to find an x satisfying the equation..." but it's challenging to do this well.

Instead, do the 'solving for x' in your scratchwork and just present the answer to the reader, as in Example Solution 2. You say 'this x will work' and then check that it works.