Proof Quiz #8 Solution

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Honor Code Rules

Proof Quizzes are open book, but are to be completed on your own without collaboration. To be specific, you may use your course notes, textbook, course website resources, course videos. You may not use the internet beyond the course websites. You may not ask anyone else for help (except your professor), including other humans, or posting/entering your question into the internet. You may not share the questions or answers with anyone else.

Have you read, understood, and followed the honor code rules above?

$$YES \quad / \quad NO$$

Please write your best written proof of the following theorem. You will be graded on logic as well as writing.

Theorem 1. *Let $f : \mathbb{Z} \to \{7x : x \in \mathbb{Z}\}$ be given by the function $f(x) = 7(x+1)$. Prove that f is bijective.*

Proof. First we show that the given function is injective. Suppose that $x_1, x_2 \in \mathbb{Z}$ satisfy $f(x_1) = f(x_2)$. Simplifying this equation, we have

$$\begin{aligned} 7(x_1+1) &= 7(x_2+1) \\ x_1 + 1 &= x_2 + 1 \\ x_1 &= x_2. \end{aligned}$$

Therefore $x_1 = x_2$. We have shown f is injective.

Next, we will show that f is surjective. Let $y \in \{7x : x \in \mathbb{Z}\}$. Then $y = 7z$ for some integer z. Let $x = z - 1$. Then x is an integer satisfying

$$f(x) = f(z-1) = 7(z-1+1) = 7z = y.$$

Hence f is surjective.

□

If you, like many students, figured out what x mapped to a given y by solving $7(x+1) = y$ for x, then you probably got $x = y/7 - 1 or (y-7)/7$. Here's a more direct way to use that to write the surjectivity part.

Alternate surjectivity portion. ⋯

Next, we will show that f is surjective. Let $y \in \{7x : x \in \mathbb{Z}\}$. Then let $x = y/7 - 1$. Since y is a multiple of 7, we see that $x \in \mathbb{Z}$. Then we verify that

$$f(x) = f(y/7-1) = 7(y/7-1+1) = y.$$

Hence f is surjective. □

Also, note that checking $y/7 - 1$ is an integer is important here.

Many students want to include their process of solving for x as part of the proof. To do this well requires carefully couching it in language that allows us to work with the equation $f(x) = y$ before we know that it has a solution. What we *cannot* do is say "Let x be such that $f(x) = y$. Then, solving the equation for x...". The reason we cannot do that is that we do not yet know (have not yet proven) that such an x exists. Also, you don't need to explain the solution process in a proof. It's enough to magically pull the answer out of a hat. :)

However, what follows is one way you could manage to include the solving process in your proof. But in the end, it's more cumbersome than my solution above.

Alternate proof of the surjectivity portion. $\cdots$

Next, we will show that f is surjective. Let $y \in \{7x : x \in \mathbb{Z}\}$. We will have shown surjectivity if we can find $x \in \mathbb{Z}$ such that $f(x) = y$. That is, we wish to find $x \in \mathbb{Z}$ such that $7(x + 1) = y$. This latter equation can be rewritten as $x = y/7 - 1$. Because y is a multiple of 7, the quantity $y/7 - 1$ is an integer. Therefore there is such an x, namely $y/7 - 1$. Hence f is surjective. $\square$