# Quiz Proof #5 Solutions

February 26, 2018

## Tools

You may call on the following definitions and lemmas if you desire.

**Definition 1.** *A graph $G$ is an ordered pair $G = (V, E)$ where $V$ is a set whose elements are called* vertices *and $E$ is a set of 2-element subsets of $V$. The elements of $E$ are called* edges.

**Definition 2.** *The* degree *of a vertex $v \in V$ is the number of edges containing that vertex.*

**Lemma 1.** *Let $a, b \in \mathbb{Z}$. If $p$ is a prime dividing $ab$, and $p$ does not divide $a$, then $p$ divides $b$.*

## Task

Write a self-contained, clearly written proof of the following fact:

**Theorem 1.** *Suppose $G$ is a graph with $N$ vertices and $M$ edges. Suppose all the vertices are of degree 3. Then 3 divides $M$.*

*Hint: First, show that $3N = 2M$.*

Proof using half-integer language:

*Proof.* Suppose $G$ is a graph with $N$ vertices and $M$ edges. Dividing each edge into two half-edges, we have $2M$ half-edges, each attached to just one vertex.

We can also count the half-edges vertex-by-vertex. Each vertex $v$ is attached to 3 half-edges. Therefore the total number of half edges is $3N$.

Since the two counts must agree, we discover that $3N = 2M$.

Therefore, 3 divides $2M$. Since 3 does not divide 2, it divides $M$ (using Lemma 1), and we are done. □

Proof using double-counting language:

*Proof.* Suppose $G$ is a graph with $N$ vertices and $M$ edges. Suppose we count the edges by counting those adjacent to each vertex $v$. If we do so, since each vertex has degree 3, we will obtain a count of $3N$. However, we will have over-counted: we will have counted each edge twice (once from each of its vertices). Therefore, the correct number of edges is $M = 3N/2$. We rewrite this as $2M = 3N$.

Therefore, 3 divides $2M$. Since 3 does not divide 2, it divides $M$ (using Lemma 1), and we are done. □