Quiz Proof #4

February 19, 2018

Theorem 1. *Suppose* $0 < x_1 < x_2 < \ldots < x_{11} < 1$ *are real numbers (listed in increasing order, and lying in the interval* $(0,1)$*). Then there is some consecutive pair within distance* $\frac{1}{10}$ *of one another, i.e. there is some* i *satisfying* $1 \leq i \leq 10$*, such that* $x_{i+1} - x_i \leq \frac{1}{10}$.

Here is a proof by contradiction:

Proof. Suppose not. Then $x_{i+1} - x_i > \frac{1}{10}$ for each i. Then

$$x_{11} - x_1 = (x_{11} - x_{10}) + (x_{10} - x_9) + \cdots + (x_2 - x_1) > 10(1/10) = 1.$$

However, both x_{11} and x_1 are in the interval $(0,1)$ so $x_{11} - x_1 < 1$. This is a contradiction. □

Here is a proof by pigeonhole:

Proof. Divide the interval $(0,1)$ into ten disjoint[1] subintervals, thus:

$$(0, 0.1], (0.1, 0.2], \ldots, (0.9, 1).$$

Since the eleven points $x_1, \ldots, x_{11}$ are in the interval $(0,1)$, each must lie in one and only one subinterval. Since there are ten intervals and eleven points, by pigeonhole principle, there are two points in one subinterval. But as the width of each subinterval is $1/10$, these two points are within distance $1/10$ of each other. □

Here is a proof using averages:

Proof. Let s denote the sum of the gap sizes between consecutive points, i.e.

$$s = (x_{11} - x_{10}) + (x_{10} - x_9) + \cdots + (x_2 - x_1).$$

There are exactly 10 gaps between 11 points. Then the average gap size is $s/10$. Since all points are in the inveral $(0,1)$, we have $s = x_{11} - x_1 < 1$. Therefore, $s/10 < 1/10$. In other words, the average gap size is less than $1/10$. Therefore, at least one gap must be less than $1/10$. □

And for good measure, here is a proof by *contrapositive*. What I'll do is *assume the conclusion fails and prove the hypothesis fails*. It is educational to compare this to the proof by contradiction.

Proof. Suppose that $x_1 < x_2 < \cdots < x_{11}$ are real numbers. Suppose that each pair is of distance greater than $1/10$. In other wrods, $x_{i+1} - x_i > 1/10$ for each $i = 1, \ldots, 10$. Then, we can compute

$$x_{11} - x_1 = (x_{11} - x_{10}) + (x_{10} - x_9) + \cdots + (x_2 - x_1) > 10(1/10) = 1.$$

Therefore the real numbers x_1 through x_{11} are not contained in the interval $(0,1)$. □

[1] Disjointness is important to the proof. Can you see why?