Proof Quiz #3

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Honor Code Rules

Proof Quizzes are open book, but are to be completed on your own without collaboration. To be specific, you may use your course notes, textbook, course website resources, course videos. You may not use the internet beyond the course websites. You may not ask anyone else for help (except your professor), including other humans, or posting/entering your question into the internet. You may not share the questions or answers with anyone else.

Have you read, understood, and followed the honor code rules above?

YES / *NO*

Please write your best written proof of the following theorem. You will be graded on logic as well as writing.

Theorem 1. *Let n be a positive integer. Let $S = \{x \in \mathbb{Z} : 1 \leq x \leq n\}$. Then the number of subsets of S having size k and including the element 1 is*

$$\frac{(n-1)!}{(n-k)!(k-1)!}.$$

Important: You may not call upon a definition of binomial coefficients, or "choose" notation, and you may not call on a known fact about the number of subsets of something. The point here is to recreate a proof from first principles, in terms of multiplication principle and overcounting.

Hints: For process, put the number 1 in the subset first, then choose some more elements, but don't forget to compensate for the overcounting you end up with. Reviewing the videos from class may help.

Proof. Let n be a positive integer and let S be as in the theorem statement.

We wish to count the number of subsets of S including 1 and having size k. Equivalently, we can consider the element 1 to be chosen first, and instead count the number of subsets of $S - \{1\}$ having size $k-1$ (i.e. choosing the rest of the subset). Note that $S - \{1\}$ is of cardinality $n-1$.

The number of ways one can arrange $k-1$ of the $n-1$ elements of $S - \{1\}$ in order is given by multiplication principle, as follows. The first item can be chosen to be one of $n-1$ items. The second is one of $n-2$ remaining items. Continuing in this manner for $k-1$ chosen items, we obtain

$$(n-1)(n-2)\cdots(n-k+1) = (n-1)!/(n-k)!.$$

Each subsets of size $k-1$ of $S - \{1\}$ can be arranged in $(k-1)!$ different ways (by a similar argument), each of which is counted individually above. Therefore, the number of subsets of size $k-1$ of $S - \{1\}$ is obtained by dividing the previous result by the overcounting factor $(k-1)!$. The result is

$$\frac{(n-1)!}{(n-k)!(k-1)!}.$$

□