Math 2001, January 18, 2018

# A very first proof

**Definition.** *An integer $n$ is called* even *if it has the form $n = 2k$ for some integer $k$.*

**Theorem.** *If $n$ is an even integer, then $n^2$ is an even integer.*

*Proof.* Suppose $n$ is an even integer.
Then $n$ has the form $n = 2k$ for some integer $k$.
Therefore, squaring both sides,

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Since $k$ is an integer, so is $2k^2$.
Therefore $n^2$ has the form $n^2 = 2\ell$, where $\ell = 2k^2$ is an integer.
Therefore, $n^2$ is even. □

## Some other phrasings

**Definition 1.** *Let $n$ be an integer. We say that $n$ is* even *if it is equal to twice some other integer.*

**Theorem 2.** *The square of an even integer is even.*

*Proof.* Assume that $n$ is an even integer.
Then we may write $n = 2k$ with $k$ being an integer.
Squaring both sides, we see that $n^2 = (2k)^2$.
We may rewrite this as $n^2 = 2(2k^2)$.
Now, we have expressed $n^2$ as twice the integer $2k^2$.
Therefore, $n^2$ is even. □

*Proof.* Let $n$ be an even integer.
Then, by the definition of an even integer, $n = 2k$ for some $k \in \mathbb{Z}$.
Squaring, we obtain $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
Note that $2k^2$ is an integer, since it is a product of integers.
Therefore $n^2 = 2r$ with $r = 2k^2 \in \mathbb{Z}$.
Hence $n^2$ is even, by the definition of an even integer. □

## The wordy proof

Let us prove the theorem together.

For the sake of argument, please imagine that you have selected an even integer, and call it $n$. It could be any even integer at all.

Now, since it is even, it must be twice another integer. Let's call the new one $k$. In other words, $n$ is twice $k$, which we can write as $n = 2k$.

Next, the equation $n = 2k$ can be squared on both sides to yield another correct equation, namely: $n^2 = (2k)^2$.

We can rewrite the right-hand side of this somewhat, so that we get $n^2 = 2(2k^2)$.

So far, what we've learned is that for whatever even $n$ you picked, there is an integer $k$ so that together they satisfy this equation, namely, $n^2 = 2(2k^2)$.

But this equation says that $n^2$ is twice $2k^2$.

And $2k^2$ is an integer, since it is a product of integers.

So $n^2$ is twice an integer.

But that means $n^2$ is even.

So, let's sum up the argument. I have described a method whereby, no matter which even integer $n$ you had in mind, I showed you how to see that $n^2$ is even.

More precisely, given that you knew what to double to obtain $n$, I was able to use that knowledge to show you what to double to obtain $n^2$. I gave you recipe for turning knowledge of $n$ into knowledge of $n^2$: if $n = 2k$ then $n^2 = 2(2k^2)$.

## An example run through

You pick 6. Six is even since it is twice 3. So we will apply the proof recipe our example, $n = 6$ and $k = 3$.

In particular, the proof says that $n^2$ should be twice $2k^2$. That is, 36 should be twice 18. It worked!