

**MATHEMATICS 2001**  
**WORKSHEET ON PIGEONHOLE PRINCIPLE**

**Prove it!** For each theorem, please give an example illustrating the theorem, if possible, before proving it. For example, for Theorem 2, choose 3 random positive integers and verify that two have the same parity.

**Theorem 1.** *Let  $n$  and  $k$  be integers. Suppose  $n$  pigeons are to be placed into  $k$  holes. Suppose  $n > k$ . Then at least one hole is shared (i.e. has more than one pigeon in it).*

*Proof by contrapositive.* Let  $n$  and  $k$  be integers. Suppose  $n$  pigeons are placed into  $k$  holes. Suppose that no holes are shared. Then, there is at least one hole per pigeon, so that  $k \geq n$ . This proves the theorem by contrapositive.  $\square$

*Proof by contradiction.* Let  $n$  and  $k$  be integers. Suppose  $n$  pigeons are placed into  $k$  holes. Suppose  $n > k$ . Suppose, for a contradiction that no holes are shared. Then, there is at least one hole per pigeon, so that  $k \geq n$ . This is a contradiction.  $\square$

*Notice the similarity between the two methods. The first is preferable.*

**Theorem 2.** *Among any 3 positive integers, there exist two of the same parity.*

*Proof.* Consider two bins, labelled “odd” and “even”. Placing each of the three integers into the bin corresponding to its parity, we are placing three items into two bins. By the pigeonhole principle, there must be at least two items in one of the bins. In other words, two of the integers have the same parity.  $\square$

**Theorem 3.** *Suppose 5 points are placed in the interior of a square with unit sides. Then some pair of these points is at distance less than or equal to  $1/\sqrt{2}$ .*

*Proof.* For concreteness, suppose the square is centered on the origin in the plane. Call the square  $S$ . Divide the square into four quadrants, i.e. the sets

$$S \cap \{(x, y) : x \geq 0, y \geq 0\}$$

$$S \cap \{(x, y) : x < 0, y \geq 0\}$$

$$S \cap \{(x, y) : x \geq 0, y < 0\}$$

$$S \cap \{(x, y) : x < 0, y < 0\}$$

Then, each point lies in one and only one quadrant. Therefore, by the pigeonhole principle, there must be two points lying in the same quadrant. But each quadrant is a box with sides  $1/2$ , hence its longest diagonal is length  $1/\sqrt{2}$ . This implies the two points contained within it can be at a distance of at most  $1/\sqrt{2}$ .  $\square$

**Theorem 4.** *Amongst any  $n$  positive integers, there exist two whose difference is divisible by  $n - 1$ .*

*Proof.* Label  $n - 1$  bins by the labels  $0, 1, \dots, n - 2$ . For each of the  $n$  positive integers, determine its remainder  $k$  upon division by  $n - 1$ . Place it into the bin labelled  $k$ . Then we are placing  $n$  integers into  $n - 1$  bins. By the pigeonhole principle, there must be two integers  $x_1$  and  $x_2$  in the same bin. But then they have the same remainder when divided by  $n - 1$ . Hence they can be expressed as  $x_1 = \ell_1(n - 1) + k$  and  $x_2 = \ell_2(n - 1) + k$ . Therefore

$$x_1 - x_2 = (\ell_1 - \ell_2)(n - 1)$$

and their difference is divisible by  $n - 1$ .  $\square$

**Theorem 5.** *Suppose  $n \geq 2$  people are in a room together. Suppose each pair is either a pair of friends or not. Then there are two people with the same number of friends.*

Note: I amended the statement to  $n \geq 2$  since it doesn't make sense if there's only one or zero people.

*Proof.* First, if two people in the room have no friends at all, then we are done. So we can assume there is at most one person in the room with no friends at all. If there is one such person, we can ignore that person and reduce the problem to the remaining people, all of whom have at least one friend. Therefore, without loss of generality, we can assume everyone has at least one friend. Label  $n$  bins by the labels  $1, \dots, n - 1$ .

Place each person into the bin labelling the number of friends he/she has. We are placing  $n$  people into  $n - 1$  bins, so by the pigeonhole principle, two people must have the same number of friends.  $\square$

**Theorem 6.** Any  $X \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$  such that  $|X| = 5$  will include two elements  $a$  and  $b$  such that  $a + b = 9$ .

*Proof.* Label four bins with the labels “1+8”, “2+7”, “3+6”, and “4+5”. These correspond to the four possible ways to build a pair of elements  $a$  and  $b$  with  $a + b = 9$ . Now, consider the five elements of the set  $X$ . Place each element  $x$  into the bin having  $x$  as one of the two digits in the label. By the pigeonhole principle, placing 5 things into 4 bins results in at least two items sharing a bin. Since the items are distinct, if there are two items in “ $x + y$ ”, then one is  $x$  and the other is  $y$ , hence we have two items in our set  $X$  which sum to 9.  $\square$

For this problem, define an *L-shaped region* of a chessboard to be a 5-square capital L shaped region, i.e. one corner square, plus two squares above it and two squares to the right.

**Theorem 7.** No matter how one colours an  $8 \times 8$  chessboard with black and white, there will always be two *L-shaped regions* that have the same colouring.

*Proof.* On an  $8 \times 8$  chessboard, there are  $6 \times 6 = 36$  locations one may place an L-shaped region (to see this, consider that the corner square of the L can be in any of the last 6 rows and first 6 columns, to allow space for the legs of the L). There are, however,  $2^5 = 32$  ways to colour an L-shaped region. Consider all 36 L-shaped regions and assign each of them to one of 32 bins based on their colouring. By pigeonhole principle, we find that two L-shaped regions must have the same colouring.  $\square$

**Theorem 8.** Let  $n$  be a natural number. Then there exist distinct natural numbers  $a$  and  $b$  such that  $n^a - n^b$  is divisible by 10.

*Proof.* Let  $n$  be a natural number. Label 10 bins by the labels 0, 1,  $\dots$ , 9. Then, for each natural number  $a$ , place it into the bin labelled by the last digit of  $n^a$ . Then, since there are infinitely many natural numbers, by pigeonhole principle, two of them are in the same bin. Thus  $n^a$  and  $n^b$  have the same last digit. In other words,  $n^a - n^b$  is divisible by 10.  $\square$

**Theorem 9.** Given five distinct lattice points in the plane, at least one of the line segments defined as joining two such points has a lattice point as a midpoint.

*Proof.* The lattice points of the plane are taken to be those with integer coordinates. Let  $(a, b)$  and  $(c, d)$  be two lattice points in the plane, i.e.  $a, b, c, d \in \mathbb{Z}$ . Their midpoint is  $(\frac{a+b}{2}, \frac{c+d}{2})$ . For this to be a lattice point (i.e. integer coordinates), it is necessary and sufficient to have that  $a$  and  $b$  have the same parity and that  $c$  and  $d$  have the same parity. Therefore, label four boxes with *(even, even)*, *(even, odd)*, *(odd, even)* and *(odd, odd)*. Place each point  $(a, b)$  into the box whose parities correspond to the parities of  $a$  and  $b$ . Then, by the pigeonhole principle, some box contains two points. Those two points are then such that their midpoint is a lattice point.  $\square$

**Card Problem.** Consider this magic trick: A magician asks an audience member to pick five cards, which are not shown to the magician. The magician’s accomplice looks at the cards, picks four of the cards, and shows these four to the magician in an order of his choosing. The magician then correctly guesses the fifth card.

Can you figure out a mathematical way to perform this trick? If so, I’ll ask you to demonstrate it on your classmates! Hint: The pigeonhole principle guarantees that such a trick is possible. But it is challenging to come up with a good way to do it.