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Proof by contradiction.

Let n and k be integers. Suppose n pigeons are placed into k holes. Suppose $n > k$. Suppose, for a contradiction that no holes are shared. Then, there is at least one hole per pigeon, so that $k \geq n$. This is a contradiction. □

Notice the similarity between the two methods. The first is preferable.

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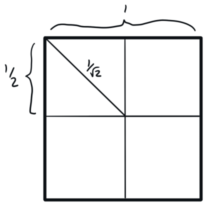
Theorem

Consider a unit square with sides of length 1*. Suppose* 5 *points are placed inside this square. Then there exist two points* x *and* y *among these five, such that the distance between* x *and* y *is less than or equal to* $1/\sqrt{2}$*.*



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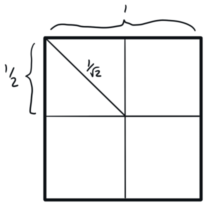

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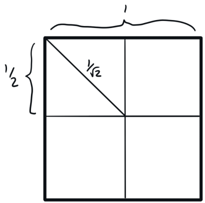

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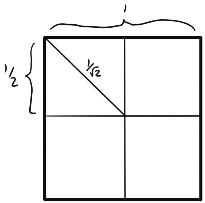

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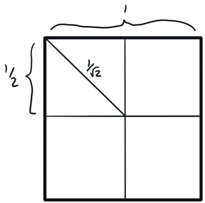

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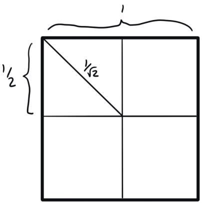

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Any $X \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ *such that* $|X| = 5$ *will include two elements* a *and* b *such that* $a + b = 9$.

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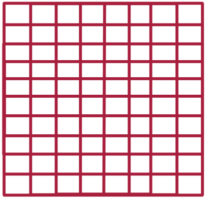

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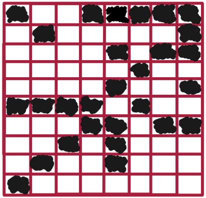

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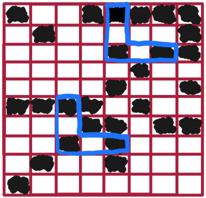

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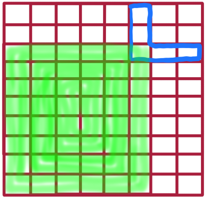

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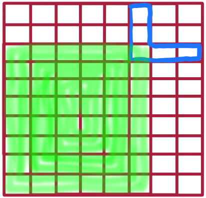
On an 8×8 chessboard, there are $6 \times 6 = 36$ locations one may place an L-shaped region (to see this, consider that the corner square of the L can be in any of the first 6 rows and first 6 columns, to allow space for the legs of the L).



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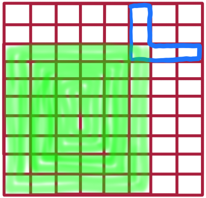

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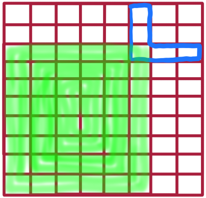

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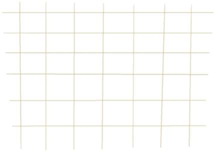

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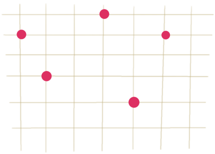
Given five distinct lattice points in the plane, at least one of the line segments defined as joining two such points has a lattice point as a midpoint.



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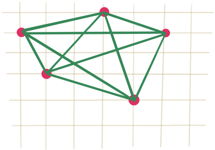

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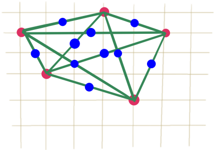

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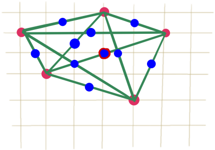

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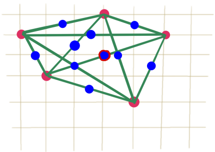

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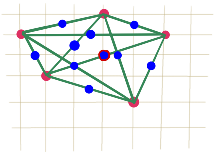

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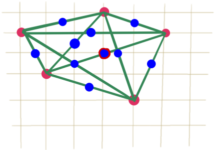

Theorem

Given five distinct lattice points in the plane, at least one of the line segments defined as joining two such points has a lattice point as a midpoint.

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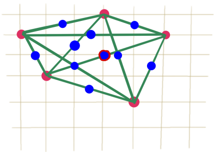

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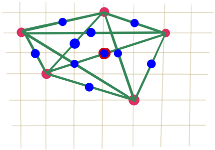

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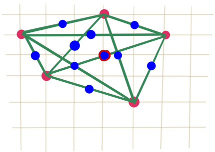

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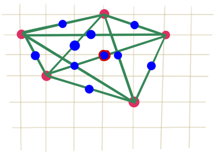
Place each point (a, b) into the box whose parities correspond to the parities of a and b.



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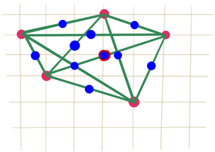

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