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Proof by contradiction.

Let n and k be integers. Suppose n pigeons are placed into k holes. Suppose $n > k$. Suppose, for a contradiction that no holes are shared. Then, there is at least one hole per pigeon, so that $k \geq n$. This is a contradiction. □

Notice the similarity between the two methods. The first is preferable.

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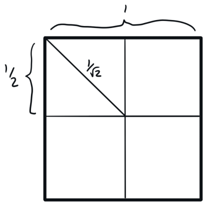
Consider two bins, labelled “odd” and “even”. Placing each of the three integers into the bin corresponding to its parity, we are placing three items into two bins. By the pigeonhole principle, there must be at least two items in one of the bins. In other words, two of the integers have the same parity. □

Theorem

Consider a unit square with sides of length 1*. Suppose* 5 *points are placed inside this square. Then there exist two points* x *and* y *among these five, such that the distance between* x *and* y *is less than or equal to* $1/\sqrt{2}$*.*



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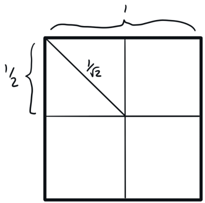

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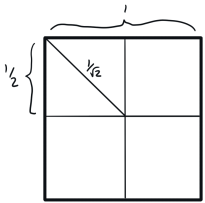

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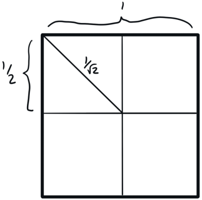

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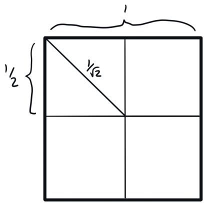

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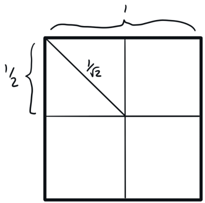

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Any $X \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ *such that* $|X| = 5$ *will include two elements* a *and* b *such that* $a + b = 9$*.*

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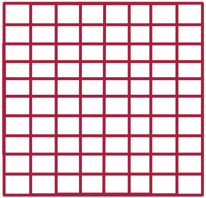

Theorem

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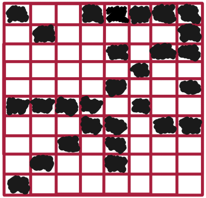

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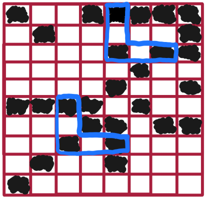

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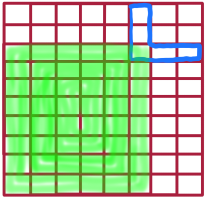

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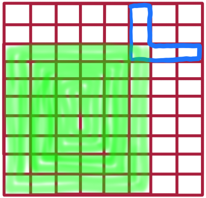
On an 8×8 chessboard, there are $6 \times 6 = 36$ locations one may place an L-shaped region (to see this, consider that the corner square of the L can be in any of the first 6 rows and first 6 columns, to allow space for the legs of the L).



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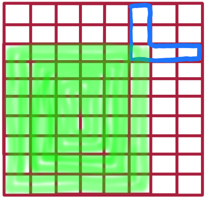

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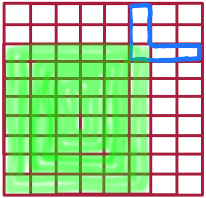

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