Pruning trees: A motivational problem for proof by induction

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Theorem

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Proof Idea:
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1. Every tree can be built out of smaller trees.
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2. The truth of the statement is preserved by this building process.
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Proof Idea:

1. Every tree can be built out of smaller trees.
2. The truth of the statement is preserved by this building process.
3. The statement holds for the smallest trees.
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Let $T$ be a tree on $n$ vertices.

Let $e$ be an edge of $T$.

If we delete $e$, we will be left with two trees. Call them $T_1$ and $T_2$.

These trees both have strictly fewer vertices than $n$. 
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- Suppose $T_1$ has $n_1$ vertices and $T_2$ has $n_2$ vertices.
- If the statement holds for the two smaller trees, then they have $n_1 - 1$ and $n_2 - 1$ edges, respectively.

So if the statement is true for the smaller trees, then it holds for $T$ also.
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- Suppose $T_1$ has $n_1$ vertices and $T_2$ has $n_2$ vertices.
- If the statement holds for the two smaller trees, then they have $n_1 - 1$ and $n_2 - 1$ edges, respectively.
- Then $T$ has $n = n_1 + n_2$ vertices and $(n_1 - 1) + (n_2 - 1) + 1 = n - 1$ edges.
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- So the theorem holds in this particular case.
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- **Base Case:** Show that the theorem is true for \( n = 1 \) (or an appropriate list of small \( n \)).
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- **Base Case:** Show that the theorem is true for \( n = 1 \) (or an appropriate list of small \( n \)).
- **Inductive Step:** Show that if the theorem is true for \( k < n \), then it is true for \( n \).
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- We will prove this by induction on the number of vertices, \( n \).
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**Theorem**

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- **Base Case:** The tree with one vertex has zero edges. Therefore the theorem holds for \( n = 1 \).
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- Then consider a tree \( T \) with \( n \) vertices. Let \( e \) be an edge of \( T \).
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- Removing \( e \) leaves two trees \( T_1 \) and \( T_2 \). Let’s suppose they have \( n_1 \) and \( n_2 \) vertices.
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- Since \( n_1, n_2 < n \), the inductive hypothesis applies to \( T_1 \) and \( T_2 \).
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- Also, $n_1 + n_2 = n$ since removing an edge does not remove any vertices.
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- Therefore $T_1$ and $T_2$ have $n_1 - 1$ edges and $n_2 - 1$ edges, respectively.
- Also, $n_1 + n_2 = n$ since removing an edge does not remove any vertices.
- Now, we count the edges of $T$:
  - $n_1 - 1$ from $T_1$
  - $n_2 - 1$ from $T_2$
  - $1$ (the edge $e$)
- Therefore $T$ has $(n_1 - 1) + (n_2 - 1) + 1 = n_1 + n_2 - 1 = n - 1$ edges.