Pruning trees: A motivational problem for proof by induction

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3. The statement holds for the smallest trees.

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- These trees both have strictly fewer vertices than n.

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- Suppose T_1 has n_1 vertices and T_2 has n_2 vertices.
- If the statement holds for the two smaller trees, then they have $n_1 - 1$ and $n_2 - 1$ edges, respectively.
- Then T has $n = n_1 + n_2$ vertices and $(n_1 - 1) + (n_2 - 1) + 1 = n - 1$ edges.

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- If the statement holds for the two smaller trees, then they have $n_1 - 1$ and $n_2 - 1$ edges, respectively.
- Then T has $n = n_1 + n_2$ vertices and $(n_1 - 1) + (n_2 - 1) + 1 = n - 1$ edges.
- So if the statement is true for the smaller trees, then it holds for T also.

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- So the theorem holds in this particular case.

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- **Inductive Step:** Show that **if** the theorem is true for $k < n$, **then** it is true for n.

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- Therefore T has $(n_1 - 1) + (n_2 - 1) + 1 = n_1 + n_2 - 1 = n - 1$ edges.