Induction

April 13, 2020

1 The Induction Format

Induction breaks one proof into two smaller proofs. For each theorem, write the **statement of** the base case and the inductive step (weak and strong versions). Don't do the proofs, just tell me what you need to prove for each piece. In fact, some of the statements are wrong!

A note on solutions: There are many ways to write a correct solution. I've tried to vary my style a little here, to give you the sense of the range of freedom possible.

1. Every positive integer is even or odd.
   (a) **Base Case:** The number 1 is even or odd.
   (b) **Inductive Step (weak):** For $n > 1$, if $n - 1$ is even or odd, then n is even or odd.
   (c) **Inductive Step (strong):** For $n > 1$, if all integers $1 \leq k < n$ are even or odd, then n is even or odd.

   (a) **Base Case:** The number 1 is even or odd.
   (b) **Inductive Step (weak):** Let $n > 1$. Suppose $n - 1$ is even or odd. Then n is even or odd.
   (c) **Inductive Step (strong):** Let $n > 1$. Suppose all integers $1 \leq k < n$ are even or odd. Then n is even or odd.

2. Any positive number of hamsters is tasty as a treat.
   (a) **Base Case:** One hamster is tasty as a treat.
   (b) **Inductive Step (weak):** Let $n \geq 1$. Suppose n hamsters is tasty as a treat. Then $n + 1$ hamsters is tasty as a treat.
   (c) **Inductive Step (strong):** Let $n > 1$. Suppose any number of hamsters between 1 and $n - 1$ inclusive, is tasty as a treat. Then n hamsters is tasty as a treat.

   (a) **Base Case:** One hamster is tasty as a treat.
   (b) **Inductive Step (weak):** Let $n \geq 1$. Suppose n hamsters is tasty as a treat. Then $n + 1$ hamsters is tasty as a treat.
   (c) **Inductive Step (strong):** Let $n > 1$. Suppose any number of hamsters between 1 and $n - 1$ inclusive, is tasty as a treat. Then n hamsters is tasty as a treat.

3. Any non-negative integer can be written as a sum of four squares.
   (a) **Base Case:** Zero can be written as a sum of four squares.
   (b) **Inductive Step (weak):** Let $n > 0$. Suppose $n - 1$ can be written as a sum of four squares. Then n can be written as a sum of four squares.
   (c) **Inductive Step (strong):** Let $n > 0$. Suppose any integer from 0 to $n - 1$ can be written as a sum of 4 squares. Then n can be written as a sum of four squares.

4. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for any positive integer n.

    (a) **Base Case:** $1 = 1(1 + 1)/2$

    (b) **Inductive Step (weak):** Let $n \geq 1$. Suppose that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Then $1 + 2 + \cdots + n + (n + 1) = \frac{(n+1)(n+2)}{2}$.

    (c) **Inductive Step (strong):** Let $n \geq 1$. Suppose that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ for all $1 \leq k < n$. Then $1 + 2 + \cdots + n + (n + 1) = \frac{(n+1)(n+2)}{2}$.

5. Any non-empty finite set of hamsters is enough hamsters to warm your heart.

    (a) **Base Case:** Any set of one hamster is enough hamsters to warm your heart.

    (b) **Inductive Step (weak):** Let n be a positive integer. If any set of n hamsters is enough hamsters to warm your heart, then any set of $n+1$ hamsters is enough hamsters to warm your heart.

    (c) **Inductive Step (strong):** Let n be a positive integer. If any set of fewer than n hamsters is enough hamsters to warm your heart, then any set of n hamsters is enough hamsters to warm your heart.

6. Any tree on $n \geq 2$ vertices has at least 2 leaves.

    (a) **Base Case:** Any tree on 2 vertices has at least 2 leaves.

    (b) **Inductive Step (weak):** Let $n > 2$. If any tree on $n - 1$ vertices has at least 2 leaves, then any tree on n vertices has at least 2 leaves.

    (c) **Inductive Step (strong):** Let $n > 2$. If any tree with at least two but fewer than n vertices has at least 2 leaves, then any tree on n vertices has at least 2 leaves.